

## LOGARITHMIC CO-HIGGS BUNDLES

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ABSTRACT. In this article we introduce a notion of logarithmic co-Higgs sheaves associated to a simple normal crossing divisor on a projective manifold, and show their existence with nilpotent co-Higgs fields for fixed ranks and second Chern classes. Then we deal with various moduli problems with logarithmic co-Higgs sheaves involved, such as coherent systems and holomorphic triples, specially over algebraic curves of low genus.

## 1. INTRODUCTION

A co-Higgs sheaf on a complex manifold  $X$  is a torsion-free coherent sheaf  $\mathcal{E}$  on  $X$  together with an endomorphism  $\Phi$  of  $\mathcal{E}$ , called a *co-Higgs field*, taking values in the tangent bundle  $T_X$  of  $X$ , i.e.  $\Phi \in H^0(\mathcal{E}nd(\mathcal{E}) \otimes T_X)$ , such that the integrability condition  $\Phi \wedge \Phi = 0$  is satisfied. When  $\mathcal{E}$  is locally free, it is a generalized vector bundle on  $X$ , considered as a generalized complex manifold and it is introduced and developed by Hitchin and Gualtieri in [16, 13]. A naturally defined stability condition on co-Higgs sheaves allows one to study their moduli spaces and Rayan and Colmenares investigate their geometry over projective spaces and a smooth quadric surface in [21, 22] and [9]. Indeed it is expected that the existence of stable co-Higgs bundles forces the position of  $X$  to be located at the lower end of the Kodaira spectrum, and Corrêa shows in [10] that a Kähler compact surface with a nilpotent stable co-Higgs bundle of rank two is uniruled up to finite étale cover. In [4, 5] the authors suggest a simple way of constructing nilpotent co-Higgs sheaves, based on Hartshorne-Serre correspondence, and obtain some (non-)existence results.

In this article we investigate the existence of nilpotent co-Higgs sheaves with a co-Higgs field vanishing in the normal direction to a given divisor of  $X$ ; for a given arrangement  $\mathcal{D}$  of smooth irreducible divisors of  $X$  with simple normal crossings, the sheaf  $T_X(-\log \mathcal{D})$  of logarithmic vector fields along  $\mathcal{D}$  is locally free and we consider a pair  $(\mathcal{E}, \Phi)$  of a torsion-free coherent sheaf  $\mathcal{E}$  and a morphism  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-\log \mathcal{D})$  with the integrability condition satisfied. The pair is called a  *$\mathcal{D}$ -logarithmic co-Higgs sheaf* and it is called 2-nilpotent if  $\Phi \circ \Phi$  is trivial. Our first result is on the existence of nilpotent  $\mathcal{D}$ -logarithmic co-Higgs sheaves of rank at least two.

**Theorem 1.1** (Propositions 3.1, 3.2 and 3.3). *Let  $X$  be a projective manifold with  $\dim(X) \geq 2$  and  $\mathcal{D} \subset X$  be a simple normal crossing divisor. For fixed  $\mathcal{L} \in \text{Pic}(X)$*

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2010 *Mathematics Subject Classification.* Primary: 14J60; Secondary: 14D20, 53D18.

*Key words and phrases.* co-Higgs bundle, double covering, nilpotent.

The first author is partially supported by MIUR and GNSAGA of INDAM (Italy). The second author is supported by Basic Science Research Program 2015-037157 through NRF funded by MEST and the National Research Foundation of Korea(KRF) 2016R1A5A1008055 grant funded by the Korea government(MSIP).

and an integer  $r \geq 2$ , there exists a 2-nilpotent  $\mathcal{D}$ -logarithmic co-Higgs sheaf  $(\mathcal{E}, \Phi)$ , where  $\Phi \neq 0$  and  $\mathcal{E}$  is reflexive and indecomposable with  $c_1(\mathcal{E}) \cong \mathcal{L}$  and  $\text{rank } \mathcal{E} = r$ .

Indeed, we can strengthen the statement of Theorem 1.1 by requiring  $\mathcal{E}$  to be locally free, in cases  $\dim(X) = 2$  or  $r \geq \dim(X)$ , due to the statement of the Hartshorne-Serre correspondence and the dimension of non-locally free locus (see Propositions 3.1 and 3.2). Moreover, in case  $\dim(X) = 2$ , we suggest an explicit number such that a logarithmic co-Higgs bundle exists for each second Chern class at least that number. We notice that the logarithmic co-Higgs sheaves constructed in Theorem 1.1 are highly unstable, which is consistent with the general philosophy on the existence of stable co-Higgs bundles (see [10, Theorem 1.1] for example).

Then we pay our attention to various different types of semistable objects involving logarithmic co-Higgs sheaves. In Section 2 we produce several examples of nilpotent semistable logarithmic co-Higgs sheaves on projective spaces and a smooth quadric surface, using a simple way of construction in [5]. Since the logarithmic co-Higgs sheaves are co-Higgs sheaves in the usual sense with an additional vanishing condition in the normal direction of divisors, so their moduli space is a closed subvariety of the moduli of the usual co-Higgs sheaves. In Section 6 we describe two moduli spaces of logarithmic co-Higgs bundles of rank two on  $\mathbb{P}^2$  in two cases.

Then in Section 7 we experiment with extensions of the notion of stability for co-Higgs sheaves and logarithmic co-Higgs sheaves. A key point for the study of moduli spaces was the introduction of parameters for the conditions of stability. We extend two of them, coherent systems and holomorphic triples, to co-Higgs sheaves. Specially in case of holomorphic triples, we show that any holomorphic triple admits the Harder-Narasimhan filtration in Corollary 7.15 and construct the moduli space of  $\nu_\alpha$ -stable  $\mathcal{D}$ -logarithmic co-Higgs triples, using Simpson's idea and quiver interpretation. We always work in cases in which there are non-trivial co-Higgs fields; so in case of dimension one we only consider projective lines and elliptic curves. We call  $\nu_\alpha$ -stability with  $\alpha \in \mathbb{R}_{>0}$ , the notion of stability for holomorphic triples. In some cases we prove that the only  $\nu_\alpha$ -stable holomorphic triples are obtained in a standard way from the same holomorphic triple taking the zero co-Higgs field (see Remark 7.23).

It is certain that a logarithmic co-Higgs field is different from a map  $\mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-D)$ , unless  $X$  is a curve. We have a glimpse of this map in Section 4 for the cases  $X = \mathbb{P}^2$  or  $\mathbb{P}^1 \times \mathbb{P}^1$ . On the contrary, in Section 5 we consider a map  $\mathcal{E} \rightarrow \mathcal{E} \otimes T_X(kD)$  with  $k > 0$ , called a meromorphic co-Higgs field, and describe semistable meromorphic co-Higgs bundles on  $\mathbb{P}^1$ .

The second author would like to thank U. Bruzzo, N. Nitsure and L. Brambila-Paz for many suggestions and interesting discussion.

## 2. DEFINITIONS AND EXAMPLES

Let  $X$  be a smooth complex projective variety of dimension  $n \geq 2$  with the tangent bundle  $T_X$ . For a fixed ample line bundle  $\mathcal{O}_X(1)$  and a coherent sheaf  $\mathcal{E}$  on  $X$ , we denote  $\mathcal{E} \otimes \mathcal{O}_X(t)$  by  $\mathcal{E}(t)$  for  $t \in \mathbb{Z}$ . The dimension of cohomology group  $H^i(X, \mathcal{E})$  is denoted by  $h^i(X, \mathcal{E})$  and we will skip  $X$  in the notation, if there is no confusion. For two coherent sheaves  $\mathcal{E}$  and  $\mathcal{F}$  on  $X$ , the dimension of  $\text{Ext}_X^1(\mathcal{E}, \mathcal{F})$  is denoted by  $\text{ext}_X^1(\mathcal{E}, \mathcal{F})$ .

To an *arrangement*  $\mathcal{D} = \{D_1, \dots, D_m\}$  of smooth irreducible divisors  $D_i$ 's on  $X$  such that  $D_i \neq D_j$  for  $i \neq j$ , we can associate the sheaf  $T_X(-\log \mathcal{D})$  of logarithmic vector fields along  $\mathcal{D}$ , i.e. it is the subsheaf of the tangent bundle  $T_X$  whose section consists of vector fields tangent to  $\mathcal{D}$ . We always assume that  $\mathcal{D}$  has simple normal crossings and so  $T_X(-\log \mathcal{D})$  is locally free. It also fits into the exact sequence [11]

$$(1) \quad 0 \rightarrow T_X(-\log \mathcal{D}) \rightarrow T_X \rightarrow \bigoplus_{i=1}^m \varepsilon_{i*} \mathcal{O}_{D_i}(D_i) \rightarrow 0,$$

where  $\varepsilon_i : D_i \rightarrow X$  is the embedding.

**Definition 2.1.** A  $\mathcal{D}$ -logarithmic co-Higgs bundle on  $X$  is a pair  $(\mathcal{E}, \Phi)$  where  $\mathcal{E}$  is a holomorphic vector bundle on  $X$  and  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-\log \mathcal{D})$  with  $\Phi \wedge \Phi = 0$ . Here  $\Phi$  is called the *logarithmic co-Higgs field* of  $(\mathcal{E}, \Phi)$  and the condition  $\Phi \wedge \Phi = 0$  is called the *integrability*.

We say that the co-Higgs field  $\Phi$  is *2-nilpotent* if  $\Phi$  is non-trivial and  $\Phi \circ \Phi = 0$ . Note that any 2-nilpotent map  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-\log \mathcal{D})$  satisfies  $\Phi \wedge \Phi = 0$  and so it is a non-zero co-Higgs structure on  $\mathcal{E}$ , i.e. a nilpotent co-Higgs structure.

Note that if  $\mathcal{D}$  is empty, then we get a usual notion of co-Higgs bundle. Indeed for each  $\mathcal{D}$ -logarithmic co-Higgs bundle we may consider a usual co-Higgs bundle by compositing the injection in (1):

$$\mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-\log \mathcal{D}) \rightarrow \mathcal{E} \otimes T_X.$$

Conversely, for a usual co-Higgs bundle  $(\mathcal{E}, \Phi)$  we may composite the surjection in (1) to have a map  $\mathcal{E} \rightarrow \bigoplus_{i=1}^m \mathcal{E} \otimes \mathcal{O}_D(D_i)$ , whose vanishing would produce a logarithmic co-Higgs structure  $\mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-\log \mathcal{D})$ . Thus our notion of logarithmic co-Higgs bundle capture the notion of a co-Higgs field  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X$  vanishing in the normal direction to the divisors in the support of  $\mathcal{D}$ ; in general it would not be asking for a map  $\varphi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-D)$  when  $\mathcal{D} = \{D\}$ . If  $\dim(X) = 1$ , then we have  $T_X(-\log \mathcal{D}) \cong T_X(-D)$ . In Section 4 we consider a few cases in which we take  $T_X(-D)$  with  $D$  smooth, instead of  $T_X(-\log \mathcal{D})$ .

**Definition 2.2.** For a fixed ample line bundle  $\mathcal{H}$  on  $X$ , a  $\mathcal{D}$ -logarithmic co-Higgs bundle  $(\mathcal{E}, \Phi)$  is  $\mathcal{H}$ -semistable (resp.  $\mathcal{H}$ -stable) if

$$\mu(\mathcal{F}) \leq (\text{resp. } <) \mu(\mathcal{E})$$

for every coherent subsheaf  $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$  with  $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes T_X(-\log \mathcal{D})$ . Recall that the slope  $\mu(\mathcal{E})$  of a torsion-free sheaf  $\mathcal{E}$  on  $X$  is defined to be  $\mu(\mathcal{E}) := \deg(\mathcal{E})/\text{rank } \mathcal{E}$ , where  $\deg(\mathcal{E}) = c_1(\mathcal{E}) \cdot \mathcal{H}^{n-1}$ . In case  $\mathcal{H} \cong \mathcal{O}_X(1)$  we simply call it semistable (resp. stable) without specifying  $\mathcal{H}$ .

**Remark 2.3.** Let  $(\mathcal{E}, \Phi)$  be a semistable  $\mathcal{D}$ -logarithmic co-Higgs bundle. For a subsheaf  $\mathcal{F} \subset \mathcal{E}$  with  $\Phi(\mathcal{F}) \subseteq \mathcal{F} \otimes T_X$ , we have

$$\mathcal{F} \otimes T_X(-\log \mathcal{D}) = (\mathcal{F} \otimes T_X) \cap (\mathcal{E} \otimes T_X(-\log \mathcal{D}))$$

and  $\text{Im}(\Phi) \subseteq \mathcal{E} \otimes T_X(-\log \mathcal{D})$ . Thus we get  $\Phi(\mathcal{F}) \subseteq \mathcal{F} \otimes T_X(-\log \mathcal{D})$  and so  $(\mathcal{E}, \Phi)$  is semistable as a usual co-Higgs bundle.

Let us denote by  $\mathbf{M}_{\mathcal{D}, X}(\chi(t))$  the moduli space of semistable  $\mathcal{D}$ -logarithmic co-Higgs bundles with Hilbert polynomial  $\chi(t)$ . It exists as a closed subscheme of  $\mathbf{M}_X(\chi(t))$  the moduli space of semistable co-Higgs bundles with the same Hilbert polynomial, since the vanishing of co-Higgs fields in the normal direction to  $\mathcal{D}$  is a

closed condition. We also denote by  $\mathbf{M}_{\mathcal{D},X}^{\circ}(\chi(t))$  the subscheme consisting of stable ones.

**Example 2.4.** Let  $X = \mathbb{P}^1$  and  $\mathcal{D} = \{p_1, \dots, p_m\}$  be a set of  $m$  distinct points on  $X$ . Then we have  $T_{\mathbb{P}^1}(-\log \mathcal{D}) \cong \mathcal{O}_{\mathbb{P}^1}(2-m)$ . Let  $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$  be a vector bundle of rank  $r \geq 2$  on  $\mathbb{P}^1$  with  $a_1 \geq \dots \geq a_r$  and  $(\mathcal{E}, \Phi)$  be a semistable  $\mathcal{D}$ -logarithmic co-Higgs bundle, i.e.  $\Phi : \mathcal{E} \rightarrow \mathcal{E}(2-m)$ . If  $a_1 = \dots = a_r$ , then the pair  $(\mathcal{E}, \Phi)$  is semistable for any  $\Phi$ . If  $m \geq 3$ , then  $\mathcal{O}_{\mathbb{P}^1}(a_1)$  would contradict the semistability of  $(\mathcal{E}, \Phi)$ , unless  $a_1 = \dots = a_r$ . If  $a_1 = \dots = a_r$  and  $m \geq 3$ , then we have  $\Phi = 0$  and so  $(\mathcal{E}, \Phi)$  is strictly semistable. Assume now that  $m \in \{0, 1, 2\}$  and then the corresponding moduli space  $\mathbf{M}_{\mathcal{D},\mathbb{P}^1}(rt+d)$  is projective and  $\mathbf{M}_{\mathcal{D},\mathbb{P}^1}^{\circ}(rt+d)$  is smooth with dimension  $(2-m)r^2 + 1$ , where  $d = r + \sum_{i=1}^m a_i$  by [17]. The case  $m = 0$  is dealt in [21, Theorem 6.1]. Now assume  $m = 1$ . Adapting the proof of [21, Theorem 6.1], we get Proposition 5.3 which says in the case  $\ell = -1$  that the existence of a map  $\Phi$  with  $(\mathcal{E}, \Phi)$  semistable implies that  $a_i \leq a_{i+1} + 1$  for all  $i$ , while conversely, if  $a_i \leq a_{i+1} + 1$  for all  $i$ , then there is a map  $\Phi$  with  $(\mathcal{E}, \Phi)$  stable and the set of all such  $\Phi$  is a non-empty open subset of the vector space  $H^0(\mathcal{E}nd(\mathcal{E})(1))$ .

Now assume that  $m = 2$  and so  $\Phi \in \text{End}(\mathcal{E})$ . If  $a_1 = \dots = a_r$ , then  $\Phi$  is given by an  $(r \times r)$ -matrix of constants. Since the matrix has an eigenvector, the pair  $(\mathcal{E}, \Phi)$  is strictly semistable for any  $\Phi$ . Now assume  $a_1 > a_r$  and let  $h$  be the maximal integer  $i$  with  $a_i = a_1$ . Write  $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{G}$  with  $\mathcal{F} := \bigoplus_{i=1}^h \mathcal{O}_{\mathbb{P}^1}(a_i)$  and  $\mathcal{G} := \bigoplus_{i=h+1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ . Since any map  $\mathcal{F} \rightarrow \mathcal{G}$  is the zero map, we have  $\Phi(\mathcal{F}) \subseteq \mathcal{F}$  for any  $\Phi : \mathcal{E} \rightarrow \mathcal{E}$  and so  $(\mathcal{E}, \Phi)$  is not semistable.

**2.1. Projective spaces.** In [5] we introduce a simple way of constructing nilpotent co-Higgs sheaves  $(\mathcal{E}, \Phi)$  of rank  $r \geq 2$ , fitting into the exact sequence

$$(2) \quad 0 \rightarrow \mathcal{O}_X^{\oplus(r-1)} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \mathcal{A} \rightarrow 0$$

for a two-codimensional locally complete intersection  $Z \subset X$  and  $\mathcal{A} \in \text{Pic}(X)$  such that  $H^0(T_X \otimes \mathcal{A}^{\vee}) \neq 0$ . We replace  $T_X$  by  $T_X(-\log \mathcal{D})$  for a simple normal crossing divisor  $\mathcal{D}$  in (2) to obtain 2-nilpotent  $\mathcal{D}$ -logarithmic co-Higgs sheaves.

**Example 2.5.** Let  $X = \mathbb{P}^n$  with  $n \geq 2$  and take  $\mathcal{D} = \{D_1, \dots, D_m\}$  with  $D_i \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ . If  $1 \leq m \leq n$ , we have  $T_{\mathbb{P}^n}(-\log \mathcal{D}) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus(m-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-m+1)}$  by [12], and in particular we have  $h^0(T_{\mathbb{P}^n}(-\log \mathcal{D})(-1)) > 0$ . Thus we may apply the proof of [5, Theorem 1.1] to get the following: here the invariant  $x_{\mathcal{E}}$  is defined to be the maximal integer  $x$  such that  $h^0(\mathcal{E}(-x)) \neq 0$ .

**Proposition 2.6.** *The set of nilpotent maps  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$  on a fixed stable reflexive sheaf  $\mathcal{E}$  of rank two on  $\mathbb{P}^n$  is an  $(n-m+1)$ -dimensional vector space only if  $c_1(\mathcal{E}) + 2x_{\mathcal{E}} = -3$ . In the other cases the set is trivial.*

**Remark 2.7.** Consider the case  $m = n+1$  in Example 2.5 with  $\bigcup_{i=1}^{n+1} D_i = \emptyset$ . Then we have  $T_{\mathbb{P}^n}(-\log \mathcal{D}) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus n}$ . Let  $\mathcal{E}$  be a reflexive sheaf of rank  $r \geq 2$  on  $\mathbb{P}^n$  with a semistable (resp. stable) logarithmic co-Higgs structure  $(\mathcal{E}, \Phi)$ . Note that if  $\Phi$  is trivial, the semistability (resp. stability) of  $(\mathcal{E}, \Phi)$  is equivalent to the semistability (resp. stability) of  $\mathcal{E}$ . Now assume  $\Phi \neq 0$ . Since  $T_{\mathbb{P}^n}(-\log \mathcal{D}) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus n}$ ,  $\mathcal{E}$  is not simple and in particular it is not stable. We claim that  $\mathcal{E}$  is semistable. If not, call  $\mathcal{G}$  be the first step of the Harder-Narasimhan filtration of  $\mathcal{E}$ . By a property of the Harder-Narasimhan filtration there is no non-zero map  $\mathcal{G} \rightarrow \mathcal{E}/\mathcal{G}$  and so no non-zero map  $\mathcal{G} \rightarrow (\mathcal{E}/\mathcal{G}) \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$ . Thus we get  $\Phi(\mathcal{G}) \subseteq \mathcal{G} \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$ ,

contradicting the semistability of  $(\mathcal{E}, \Phi)$ . Now assume  $n = 2$  and take  $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^2}$  in (2) with  $\deg(Z) \geq r - 1$ . Then we get many strictly semistable and indecomposable vector bundles  $\mathcal{E}$  with  $\Phi \neq 0$  and 2-nilpotent.

**Example 2.8.** Let  $X = \mathbb{P}^2$  and take  $\mathcal{D} = \{D\}$  with  $D$  a smooth conic. Since  $h^0(T_{\mathbb{P}^2}) = 8$  and  $h^0(\mathcal{O}_D(D)) = h^0(\mathcal{O}_D(2)) = 5$ , we have  $h^0(T_{\mathbb{P}^2}(-\log \mathcal{D})) > 0$  from (1). By taking  $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^2}$  in [5, Equation (1) of Condition 2.2], we get a strictly semistable logarithmic co-Higgs bundle  $(\mathcal{E}, \Phi)$  with a non-zero co-Higgs field  $\Phi$ , where  $\mathcal{E}$  is strictly semistable of any arbitrary rank  $r \geq 2$  with any non-negative integer  $c_2(\mathcal{E}) = \deg(Z)$ . Moreover, for any integer  $c_2(\mathcal{E}) \geq r - 1$  we may find an indecomposable one.

**Example 2.9.** Let  $X \subset \mathbb{P}^{n+1}$  be a smooth quadric hypersurface. Let  $D \subset X$  be a smooth hyperplane section of  $X$  with  $H \subset \mathbb{P}^{n+1}$  the hyperplane such that  $D = X \cap H$  and take  $\mathcal{D} = \{D\}$ . If  $p \in \mathbb{P}^{n+1}$  is the point associated to  $H$  by the isomorphism between  $\mathbb{P}^{n+1}$  and its dual induced by an equation of  $X$ , then we have  $p \notin X$  since  $X$  is smooth. Letting  $\pi_p : X \rightarrow \mathbb{P}^n$  denote the linear projection from  $p$ , we have  $T_X(-\log \mathcal{D}) \cong \pi_p^*(\Omega_{\mathbb{P}^n}^1(2))$  by [3, Corollary 4.6]. Since  $\Omega_{\mathbb{P}^n}^1(2)$  is globally generated, so is  $T_X(-\log \mathcal{D})$  and in particular  $H^0(T_X(-\log \mathcal{D})) \neq 0$ . By taking  $\mathcal{A} \cong \mathcal{O}_X$  in [5, Equation (1) of Condition 2.2], we get a strictly semistable logarithmic co-Higgs bundle  $(\mathcal{E}, \Phi)$  with a non-zero co-Higgs field  $\Phi$ , where  $\mathcal{E}$  is strictly semistable of any arbitrary rank  $r \geq 2$ .

**2.2. Smooth quadric surfaces.** Let  $X = \mathbb{P}^1 \times \mathbb{P}^1$  be a smooth quadric surface and we may assume for a vector bundle  $\mathcal{E}$  of rank two that

$$\det(\mathcal{E}) \in \{\mathcal{O}_X, \mathcal{O}_X(-1, 0), \mathcal{O}_X(0, -1), \mathcal{O}_X(-1, -1)\}.$$

The case of the usual co-Higgs bundle with  $\mathcal{D} = \emptyset$  is done in [9, Theorem 4.3]. We assume either

- (i)  $\mathcal{D} \in \{|\mathcal{O}_X(1, 0)|, |\mathcal{O}_X(2, 0)|, |\mathcal{O}_X(0, 1)|, |\mathcal{O}_X(0, 2)|\}$ , or
- (ii)  $\mathcal{D} = L \cup R$  with  $L \in |\mathcal{O}_X(1, 0)|$  and  $R \in |\mathcal{O}_X(0, 1)|$ .

In the latter case  $T_X(-\log \mathcal{D})$  fits into the exact sequence

$$(3) \quad 0 \rightarrow T_X(-\log \mathcal{D}) \rightarrow \mathcal{O}_X(2, 0) \oplus \mathcal{O}_X(0, 2) \rightarrow \mathcal{O}_L \oplus \mathcal{O}_R \rightarrow 0,$$

because  $\mathcal{O}_L(L) \cong \mathcal{O}_L$ ,  $\mathcal{O}_R(R) \cong \mathcal{O}_R$  and  $T_X \cong \mathcal{O}_X(2, 0) \oplus \mathcal{O}_X(0, 2)$ . In particular, we have  $h^0(T_X(-\log \mathcal{D})(i, j)) > 0$  for all  $(i, j) \in \{(0, 0), (-1, 0), (0, -1)\}$ . We may also consider the following cases:

- (iii)  $\mathcal{D} = L \cup L' \cup R$  with  $L, R$  as above and  $L \neq L' \in |\mathcal{O}_X(1, 0)|$ ; we still have  $h^0(T_X(-\log \mathcal{D})(i, j)) > 0$  for  $(i, j) \in \{(0, 0), (0, -1)\}$ .
- (iv)  $\mathcal{D} = L \cup L' \cup R \cup R'$  with  $L, L', R$  as above and  $R \neq R' \in |\mathcal{O}_X(0, 1)|$ .

Indeed, if  $\mathcal{D}$  consists of  $a$  lines in  $|\mathcal{O}_X(1, 0)|$  and  $b$  lines in  $|\mathcal{O}_X(0, 1)|$ , then we have  $T_X(-\log \mathcal{D}) \cong \mathcal{O}_X(2 - a, 0) \oplus \mathcal{O}_X(0, 2 - b)$  by [3, Proposition 6.2].

Assume that  $\mathcal{E}$  fits into the following exact sequence as in [9, Equation (3.1)]

$$(4) \quad 0 \rightarrow \mathcal{O}_X(r, d) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(r', d') \otimes \mathcal{I}_Z \rightarrow 0,$$

where  $Z \subset X$  is a zero-dimensional scheme,  $\det(\mathcal{E}) \cong \mathcal{O}_X(r + r', d + d')$  and  $c_2(\mathcal{E}) = \deg(Z) + rd' + r'd$ . Note that any logarithmic co-Higgs bundle is co-Higgs in the usual sense and so the set of all  $(c_1, c_2)$  allowed for  $\mathcal{D}$  is contained in the one allowed for  $\mathcal{D} = \emptyset$ . In particular, if we are concerned only in  $\mathcal{O}_X(1, 1)$ -semistability, the possible pairs  $(c_1, c_2)$  are contained in the one described in [9,

Theorem 4.3]. Moreover, any existence for the case  $\mathcal{D} = L \cup R$  implies the existence for  $\mathcal{D} \in \{|\mathcal{O}_X(1, 0)|, |\mathcal{O}_X(0, 1)|\}$ .

(a) First assume  $\det(\mathcal{E}) \cong \mathcal{O}_X$  and we prove the existence for  $c_2 \geq 0$ . In this case we take  $r = d = r' = d' = 0$  and the 2-nilpotent co-Higgs structure induced by  $\mathcal{I}_Z \rightarrow T_X(-\log \mathcal{D})$ , i.e. by a non-zero section of  $T_X(-\log \mathcal{D})$ . This construction gives  $(\mathcal{E}, \Phi)$  with  $\mathcal{E}$  strictly semistable for any polarization.

(b) Assume  $\det(\mathcal{E}) \cong \mathcal{O}_X(-1, 0)$  by symmetry and see the existence for  $c_2 \geq 0$ . In case  $h^0(T_X(-\log \mathcal{D})(-1, 0)) > 0$ , we take  $(r, r', d, d') = (-1, 0, 0, 0)$  and  $\Phi$  induced by a non-zero map  $\mathcal{I}_Z \rightarrow T_X(-\log \mathcal{D})(-1, 0)$ . Then  $\mathcal{E}$  is stable for every polarization, unless  $Z = \emptyset$  and  $\mathcal{E}$  splits, because  $Z \neq \emptyset$  would imply  $h^0(\mathcal{E}) = 0$ ; even when  $Z = \emptyset$  and so  $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-1, 0)$ , the pair  $(\mathcal{E}, \Phi)$  is stable for every polarization.

(c) Assume  $\det(\mathcal{E}) \cong \mathcal{O}_X(-1, -1)$  and take  $(r, d) = (-1, 0)$  and  $(r', d') = (0, -1)$  with  $\mathcal{D} \in |\mathcal{O}_X(1, 0)|$ . Then we have  $h^0(T_X(-\log \mathcal{D})(-1, 1)) > 0$  and  $c_2(\mathcal{E}) = \deg(Z) + 1$ . We get that  $\mathcal{E}$  is semistable with respect to  $\mathcal{O}_X(1, 1)$ .

### 3. EXISTENCE

**Proposition 3.1.** *Assume  $\dim(X) = 2$  and let  $\mathcal{D} \subset X$  be a simple normal crossing divisor. For fixed  $\mathcal{L} \in \text{Pic}(X)$  and an integer  $r \geq 2$ , there exists an integer  $n = n_{X, \mathcal{D}}(\mathcal{L}, r)$  such that for all integers  $c_2 \geq n$  there is a 2-nilpotent  $\mathcal{D}$ -logarithmic co-Higgs bundle  $(\mathcal{E}, \Phi)$  with  $\Phi \neq 0$ , where  $\mathcal{E}$  is an indecomposable vector bundle of rank  $r$  with Chern classes  $c_1(\mathcal{E}) \cong \mathcal{L}$  and  $c_2(\mathcal{E}) = c_2$ .*

*Proof.* Fix a very ample  $\mathcal{R} \in \text{Pic}(X)$  such that

- $h^0(\omega_X \otimes (\mathcal{L}^{\otimes(r-1)} \otimes \mathcal{R}^{\otimes r})^\vee) = 0$ ;
- $h^0(T_X(-\log \mathcal{D}) \otimes \mathcal{L}^{\otimes(r-1)} \otimes \mathcal{R}^{\otimes r}) > 0$ ;
- $\mathcal{L} \otimes \mathcal{R}$  is spanned.

Set

$$n = n_{X, \mathcal{D}}(r, \mathcal{L}) := r - (r-1)(r-2)\mathcal{L}^2 - (r-1)^2\mathcal{R}^2 - (2r-3)(r-1)\mathcal{L} \cdot \mathcal{R}.$$

For each  $c_2 \geq n$ , let  $S \subset X$  be a union of general  $(c_2 + r - n)$  points and consider a general extension

$$0 \rightarrow (\mathcal{L} \otimes \mathcal{R})^{\oplus(r-1)} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_S \otimes (\mathcal{L}^{\otimes(r-2)} \otimes \mathcal{R}^{\otimes(r-1)})^\vee \rightarrow 0.$$

From the choice of  $\mathcal{R}$  the Cayley-Bacharach condition is satisfied and so  $\mathcal{E}$  is locally free with  $c_1(\mathcal{E}) \cong \mathcal{L}$  and  $c_2(\mathcal{E}) = c_2$ . Now from a non-zero section in  $H^0(T_X(-\log \mathcal{D}) \otimes \mathcal{L}^{\otimes(r-1)} \otimes \mathcal{R}^{\otimes r})$  we have a non-zero map  $\varphi : \mathcal{I}_S \otimes (\mathcal{L}^{\otimes(r-2)} \otimes \mathcal{R}^{\otimes(r-1)})^\vee \rightarrow \mathcal{L} \otimes \mathcal{R} \otimes T_X(-\log \mathcal{D})$ , inducing a non-zero map  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-\log \mathcal{D})$  that is 2-nilpotent and so integrable.

Thus to complete the proof it is sufficient to prove that  $\mathcal{E}$  is indecomposable for a suitable  $\mathcal{R}$ . Assume  $\mathcal{E} \cong \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$  with  $k \geq 2$  and each  $\mathcal{E}_i$  indecomposable and locally free of positive rank. Since  $\mathcal{R}$  is very ample and  $\mathcal{L} \otimes \mathcal{R}$  is spanned, the image of the evaluation map  $H^0(\mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  is isomorphic to  $(\mathcal{L} \otimes \mathcal{R})^{\oplus(r-1)}$  and its cokernel is isomorphic to  $\mathcal{I}_S \otimes (\mathcal{L}^{\otimes(r-2)} \otimes \mathcal{R}^{\otimes(r-1)})^\vee$ . Thus, up to a permutation of the factors, we have  $(\mathcal{L} \otimes \mathcal{R})^{\oplus(r-1)} \cong \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k-1} \oplus \mathcal{F}$  with  $\mathcal{F}$  a vector bundle and  $\mathcal{E}_k/\mathcal{F} \cong \mathcal{I}_S \otimes (\mathcal{L}^{\otimes(r-2)} \otimes \mathcal{R}^{\otimes(r-1)})^\vee$ . Since  $\mathcal{E}_1$  is indecomposable, we get that  $\mathcal{E}_1 \cong \mathcal{L} \otimes \mathcal{R}$ . But since  $\sharp(S) \geq r$ , we have  $\text{ext}_X^1(\mathcal{I}_S \otimes (\mathcal{L}^{\otimes(r-2)} \otimes \mathcal{R}^{\otimes(r-1)})^\vee, \mathcal{O}_X) \geq r$  and so we may choose  $\mathcal{E}$  so that  $\mathcal{L} \otimes \mathcal{R}$  is not a factor of  $\mathcal{E}$ .  $\square$

**Proposition 3.2.** *Assume  $n = \dim(X) \geq 3$  and let  $\mathcal{D} \subset X$  be a simple normal crossing divisor. For a fixed  $\mathcal{L} \in \text{Pic}(X)$  and an integer  $r \geq n$ , there exists a 2-nilpotent  $\mathcal{D}$ -logarithmic co-Higgs bundle  $(\mathcal{E}, \Phi)$ , where  $\mathcal{E}$  is an indecomposable vector bundle of rank  $r$  on  $X$  with  $\det(\mathcal{E}) \cong \mathcal{L}$ .*

*Proof.* We first assume that  $\mathcal{L}^\vee$  is very ample with

- $h^1(\mathcal{L}^\vee) = h^2(\mathcal{L}^\vee) = 0$ , where we use the assumption  $n \geq 3$ ;
- $h^0(\mathcal{L}^\vee) \geq r - 1$  and  $h^0(\mathcal{L}^\vee \otimes T_X(-\log \mathcal{D})) > 0$ .

Fix a very ample line bundle  $\mathcal{H}$  on  $X$  such that  $h^0(\mathcal{H}^\vee \otimes \mathcal{L}^\vee) = h^1((\mathcal{H}^\vee)^{\otimes 2} \otimes \mathcal{L}^\vee) = 0$ , e.g. by taking  $\mathcal{H} \cong (\mathcal{L}^\vee)^{\otimes 2}$  and applying Kodaira's vanishing. Let  $Y \subset X$  be a general complete intersection of two elements of  $|\mathcal{H}|$  and then  $Y$  is a non-empty connected manifold of codimension 2 with normal bundle  $N_Y$ , isomorphic to  $\mathcal{H}_{|Y}^{\oplus 2}$ . The line bundle  $\mathcal{R} := \wedge^2 N_Y \otimes \mathcal{L}_{|Y}^\vee \cong (\mathcal{H}^{\otimes 2} \otimes \mathcal{L}^\vee)_{|Y}$  is a very ample line bundle on  $Y$  and we have  $h^0(Y, \mathcal{R}) \geq h^0(Y, (\mathcal{L}^\vee)_{|Y})$ . From the exact sequence

$$0 \rightarrow (\mathcal{H}^\vee)^{\otimes 2} \rightarrow (\mathcal{H}^\vee)^{\oplus 2} \rightarrow \mathcal{I}_Y \rightarrow 0$$

we get  $h^0(\mathcal{I}_Y \otimes \mathcal{L}^\vee) = 0$  and so  $h^0(Y, \mathcal{R}) \geq h^0(Y, (\mathcal{L}^\vee)_{|Y}) \geq r - 1$ . Since  $\mathcal{R}$  is spanned and  $\dim(Y) = n - 2$ , a general  $(n - 1)$ -dimensional linear subspace  $V \subset H^0(Y, \mathcal{R})$  spans  $\mathcal{R}$ . Hence there are linearly independent sections  $s_1, \dots, s_{r-1}$  of  $H^0(Y, \mathcal{R})$  spanning  $\mathcal{R}$ . Since  $H^2(\mathcal{L}^\vee) = 0$ , by the Hartshorne-Serre correspondence the sections  $s_1, \dots, s_{r-1}$  give a vector bundle  $\mathcal{E}$  of rank  $r$  fitting into an exact sequence (see [2, Theorem 1.1])

$$0 \rightarrow \mathcal{O}_X^{\oplus(r-1)} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Y \otimes \mathcal{L} \rightarrow 0.$$

In particular we have  $\det(\mathcal{E}) \cong \mathcal{L}$ . Any non-zero section of  $H^0(\mathcal{L}^\vee \otimes T_X(-\log \mathcal{D}))$  gives a 2-nilpotent logarithmic co-Higgs structures on  $\mathcal{E}$  with  $\Phi \neq 0$ . Now it remains to show that  $\mathcal{E}$  is indecomposable. Assume  $\mathcal{E} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$  with  $\mathcal{G}_i$  non-zero. Let  $\mathcal{G}'_i$  be the image of the evaluation map  $H^0(\mathcal{G}_i) \otimes \mathcal{O}_X \rightarrow \mathcal{G}_i$  for  $i = 1, 2$ . Since  $\mathcal{L}^\vee$  is very ample, we have  $h^0(\mathcal{E}) = r - 1$  and the image of the evaluation map  $H^0(\mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  is isomorphic to  $\mathcal{O}_X^{\oplus(r-1)}$  and so  $\mathcal{G}'_1 \oplus \mathcal{G}'_2 \cong \mathcal{O}_X^{\oplus(r-1)}$ . In particular, we have  $\mathcal{G}_i \cong \mathcal{G}'_i$  for some  $i$  and so at least one of the factors of  $\mathcal{E}$  is trivial. Set  $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{F}$  with  $\text{rank}(\mathcal{F}) = r - 1$ . By [2, Theorem 1.1] the bundle  $\mathcal{F}$  comes from  $u_1, \dots, u_{r-2} \in H^0(Y, \mathcal{R})$  and so  $\mathcal{E}$  is induced by the sections  $u_1, \dots, u_{r-2}, 0$ . Since  $H^1(\mathcal{L}^\vee) = 0$ , the uniqueness part of [2, Theorem 1.1] gives that  $s_1, \dots, s_{r-1}$  generate the linear subspace of  $H^0(Y, \mathcal{R})$  spanned by  $u_1, \dots, u_{r-2}$  and so they are not linearly independent, a contradiction.

Now we drop any assumption on  $\mathcal{L}$ . Take an integer  $m \gg 0$  and set  $\mathcal{L}' := \mathcal{L} \otimes (\mathcal{H}^\vee)^{\otimes(mr)}$ . Then we get that  $(\mathcal{L}')^\vee$  is very ample and  $H^2((\mathcal{L}')^\vee) = 0$ . By the first part there is  $(\mathcal{E}', \Phi')$  with  $\det(\mathcal{E}') \cong \mathcal{L}'$ . We may take  $\mathcal{E} := \mathcal{E}' \otimes (\mathcal{H}^\vee)^{\otimes m}$  and let  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-\log \mathcal{D})$  be the non-zero map induced by  $\Phi'$ .  $\square$

Allowing non-locally free sheaves, we may extend Proposition 3.2 to all ranks at least two in the following way.

**Proposition 3.3.** *Under the same assumption as in Proposition 3.2 with  $2 \leq r \leq n - 1$ , there exists a 2-nilpotent  $\mathcal{D}$ -logarithmic co-Higgs reflexive sheaf  $(\mathcal{E}, \Phi)$ , where  $\mathcal{E}$  is indecomposable of rank  $r$  with  $\det(\mathcal{E}) \cong \mathcal{L}$  and non-locally free locus of dimension at most  $(n - r - 1)$ .*

*Proof.* We follow the proof of Proposition 3.2. We first assume that  $\mathcal{L}^\vee$  is very ample and take  $(\mathcal{H}, Y, \mathcal{R})$  as in the proof of Proposition 3.2. Since  $r \geq 2$ , we may find  $r - 1$  elements  $s_1, \dots, s_{r-1} \in H^0(Y, \mathcal{R})$  spanning  $\mathcal{R}$  outside a subset  $T$  of  $Y$  with  $\dim(T) \leq \dim(Y) - r + 1 = n - r - 1$ . By [15], the sections  $s_1, \dots, s_{r-1}$  give a reflexive sheaf  $\mathcal{E}$  of rank  $r$  on  $X$  with  $\det(\mathcal{E}) \cong \mathcal{L}$  and  $\mathcal{E}$  locally free outside  $T$ .

The reduction to the case in which  $\mathcal{L}^\vee$  is very ample can be done, using the argument in the proof of Proposition 3.2.  $\square$

#### 4. VANISHING ALONG DIVISORS

As observed, the notion of logarithmic co-Higgs bundle is not asking for a map  $\varphi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X(-D)$  if  $\dim(X) \geq 2$ . In this section we study vector bundles of rank two on a projective plane and a smooth quadric surface with sections in  $H^0(\mathcal{E}nd(\mathcal{E}) \otimes T_X(-D))$ .

**4.1. Projective plane.** Let  $X = \mathbb{P}^2$  and take  $D \in |\mathcal{O}_{\mathbb{P}^2}(1)|$  a projective line. Then we have  $T_{\mathbb{P}^2}(-D) = T_{\mathbb{P}^2}(-1)$  and so  $h^0(T_{\mathbb{P}^2}(-D)) = 3$ . We may give a 2-nilpotent co-Higgs structure on a vector bundle  $\mathcal{E}$  of rank 2 fitting into the exact sequence

$$(5) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \rightarrow 0$$

from a non-zero section in  $H^0(T_{\mathbb{P}^2}(-D))$ . Thus there exists a strictly semistable co-Higgs bundle of rank two for all  $c_2 \geq 0$ , which is indecomposable for  $c_2 > 0$ . Indeed for any such bundles with positive  $c_2$  we have a three-dimensional vector space of 2-nilpotent co-Higgs structures. On the contrary we have some results on non-existence of co-Higgs bundles on projective spaces in [5, Section 3]. Applying the same argument to  $T_{\mathbb{P}^n}(-1)$ , we get the following, as in Proposition 2.6

**Proposition 4.1.** *If  $\mathcal{E}$  is a stable reflexive sheaf of rank two on  $\mathbb{P}^n$  with  $n \geq 2$ , then any nilpotent map  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_{\mathbb{P}^n}(-1)$  is trivial.*

**4.2. Quadric surface.** Let  $X = \mathbb{P}^1 \times \mathbb{P}^1$  and take  $D \in |\mathcal{O}_X(1, 0)|$ ; by symmetry the case  $D \in |\mathcal{O}_X(0, 1)|$  is similar. We have  $T_X(-D) \cong \mathcal{O}_X(1, 0) \oplus \mathcal{O}_X(-1, 2)$ .

(a) In case  $\det(\mathcal{E}) \cong \mathcal{O}_X$  we prove the existence for  $c_2 \geq 0$ . By taking  $r = d = r' = d' = 0$ , we obtain a 2-nilpotent co-Higgs structure induced by  $\mathcal{I}_Z \rightarrow T_X(-D)$ , i.e. by a non-zero section of  $T_X(-D)$ . This construction gives  $(\mathcal{E}, \Phi)$  with  $\mathcal{E}$  strictly semistable for any polarization.

(b) In case  $\det(\mathcal{E}) \cong \mathcal{O}_X(-1, 0)$  we also see the existence for  $c_2 \geq 0$ . Since  $h^0(T_X(-D)(-1, 0)) > 0$ , we take  $(r, r', d, d') = (-1, 0, 0, 0)$  and  $\Phi$  induced by a non-zero map  $\mathcal{I}_Z \rightarrow T_X(-D)(-1, 0)$ . Then  $\mathcal{E}$  is stable for every polarization, unless  $Z = \emptyset$  and  $\mathcal{E}$  splits, because  $Z \neq \emptyset$  would imply  $h^0(\mathcal{E}) = 0$ ; even when  $Z = \emptyset$  and so  $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-1, 0)$ , the pair  $(\mathcal{E}, \Phi)$  is stable for every polarization.

(c) Assume  $\det(\mathcal{E}) \cong \mathcal{O}_X(-1, -1)$  and take  $(r, d) = (-1, 0)$  and  $(r', d') = (0, -1)$  with  $D \in |\mathcal{O}_X(1, 0)|$ . Note that  $h^0(T_X(-D)(-1, 1)) > 0$  and  $c_2(\mathcal{E}) = \deg(Z) + 1$ . Then we get that  $\mathcal{E}$  is semistable with respect to  $\mathcal{O}_X(1, 1)$ .

**Remark 4.2.** (1) It is likely that we may not apply our method of construction of 2-nilpotent co-Higgs structure to the case when  $\det(\mathcal{E}) \cong \mathcal{O}_X(0, -1)$ , because it requires a non-zero section in  $h^0(T_X(-D)(-1, 0))$ , which is trivial.

(2) Take  $\mathcal{D} = L \cup R$  with  $L, R \in |\mathcal{O}_X(1, 0)|$  and  $L \neq R$ ; the case with  $L, R \in |\mathcal{O}_X(0, 1)|$  is similar. Then the existence for the case  $c_1(\mathcal{E}) = \mathcal{O}_X(0, 0)$  can be done for any  $c_2 \geq 0$  as above.



## 5. EXTENSION OF CO-HIGGS BUNDLES

Fix an ample line bundle  $\mathcal{H}$  on  $X$  and a vector bundle  $\mathcal{G}$ . Then we may define  $\mathcal{H}$ -(semi)stability for a pair  $(\mathcal{E}, \Phi)$  with  $\mathcal{E}$  a torsion-free sheaf and  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{G}$ , similarly as in Definition 2.2 with  $\mathcal{G}$  instead of  $T_X(-\log \mathcal{D})$ . Then the definition of (logarithmic) co-Higgs bundle is obtained by taking  $\mathcal{G} \in \{T_X, T_X(-\log \mathcal{D}), T_X(-D)\}$  with the integrability condition  $\Phi \wedge \Phi = 0$ . Note that it is enough to check the integrability condition on a non-empty open subset  $U$  of  $X$ .

**Definition 5.1.** Fix an effective divisor  $D \subset X$  and a positive integer  $k$ , for which we take  $\mathcal{G} := T_X(kD)$ . A pair  $(\mathcal{E}, \Phi)$  is called a *meromorphic co-Higgs sheaf* with poles of order at most  $k$  contained in  $D$ , if it satisfies the integrability condition on  $U := X \setminus D$ .

Via the inclusion  $T_X \hookrightarrow T_X(kD)$  induced by a section of  $\mathcal{O}_X(kD)$  with  $kD$  as its zeros, we see that any co-Higgs sheaf is also a meromorphic co-Higgs for any  $k$  and  $D$ . A meromorphic co-Higgs sheaf with poles contained in  $D$  induces an ordinary co-Higgs sheaf  $(\mathcal{F}, \varphi)$  on the non-compact manifold  $U$  and our definition of meromorphic co-Higgs sheaves captures the extension of  $(\mathcal{F}, \varphi)$  to  $X$  with at most poles on  $D$  of order at most  $k$ .

**Remark 5.2.** We may generalize the definition of a meromorphic co-Higgs sheaf as follows: take  $D = \cup_{i=1}^s D_i$  with each  $D_i$  irreducible and consider  $\sum_{i=1}^s k_i D_i$ ,  $k_i$  a positive integer, instead of  $kD$ . Then we get the co-Higgs sheaves  $(\mathcal{F}, \varphi)$  on  $X \setminus D$ , which extends meromorphically to  $X$  with poles of order at most  $k_i$  on each  $D_i$ .

Our method used in constructing 2-nilpotent co-Higgs sheaves (see [5, Condition 2.2]) can be applied to construct 2-nilpotent meromorphic co-Higgs sheaves, if  $h^0(T_X(kD)) > 0$ ; we may easily check when the construction gives locally free ones. In the set-up of Sections 2.2 and 4 we immediately see how to construct examples filling in several Chern classes.

Assume that  $\dim(X) = 1$  and let  $D = p_1 + \cdots + p_s$  be  $s$  distinct points on  $X$ . Set  $\ell := \deg(\sum_{i=1}^s k_i p_i)$  and  $r := \text{rank}(\mathcal{E})$ . We adapt the proof of [21, Theorem 6.1] with only very minor modifications to prove the following result. To cover the case needed in Example 2.4 we allow as  $\ell$  an integer at least  $-1$ .

**Proposition 5.3.** *Let  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$  be a vector bundle of rank  $r \geq 2$  on  $\mathbb{P}^1$  with  $a_1 \geq \cdots \geq a_r$ .*

- (i) *If  $(\mathcal{E}, \Phi)$  is semistable with a map  $\Phi : \mathcal{E} \rightarrow \mathcal{E}(2 + \ell)$ , then we have  $a_{i+1} \geq a_i - \ell - 2$  for each  $i \leq r - 1$ .*
- (ii) *Conversely, if  $a_{i+1} \geq a_i - \ell - 2$  for each  $i \leq r - 1$ , then there is a map  $\Phi : \mathcal{E} \rightarrow \mathcal{E}(2 + \ell)$  such that no proper subbundle  $\mathcal{F} \subset \mathcal{E}$  satisfies  $\Phi(\mathcal{F}) \subseteq \mathcal{F}(2 + \ell)$ , and in particular  $(\mathcal{E}, \Phi)$  is stable. The set of all such  $\Phi$  is non-empty open subset of the vector space  $H^0(\text{End}(\mathcal{E})(2 + \ell))$ .*

*Proof.* Assume the existence of an integer  $i$  such that  $a_{i+1} \leq a_i - \ell - 3$  and take  $\Phi : \mathcal{E} \rightarrow \mathcal{E}(2 + \ell)$ . Set  $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$  with  $\mathcal{F} := \oplus_{j=1}^i \mathcal{O}_{\mathbb{P}^1}(a_j)$  and  $\mathcal{G} := \oplus_{j=i+1}^r \mathcal{O}_{\mathbb{P}^1}(a_j)$ . Since any map  $\mathcal{F} \rightarrow \mathcal{G}(2 + \ell)$  is the zero map, we have  $\Phi(\mathcal{F}) \subseteq \mathcal{F}(2 + \ell)$  and so  $(\mathcal{E}, \Phi)$  is not semistable.

Now assume  $a_{i+1} \geq a_i - \ell - 2$  for all  $i$ . Write  $\Phi$  as an  $(r \times r)$ -matrix  $B$  with entries  $b_{i,j} \in \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(a_i), \mathcal{O}_{\mathbb{P}^1}(a_j + 2 + \ell))$ . For fixed homogeneous coordinates  $z_0, z_1$  on  $\mathbb{P}^1$  with  $\infty = [1 : 0]$  and  $0 = [0 : 1]$ , see a homogeneous polynomial of

degree  $d$  in the variables  $z_0, z_1$  as a polynomial of degree at most  $d$  in the variable  $z := z_0/z_1$ . Take

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & z \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

so that  $b_{i,j} = 0$  unless either  $(i,j) = (1,r)$  or  $j = i+1$ ; we take  $b_{i,i+1} = 1$  for all  $i$ , i.e. the elements of  $\mathbb{C}[z]$  associated to  $z_1^{a_{i+1}-a_i+2+\ell}$ , and  $b_{1,r} = z$ , the element of  $\mathbb{C}[z]$  associated to  $z_0 z_1^{a_r-a_1+1+\ell}$ . Then there is no proper subbundle  $\mathcal{F} \subset \mathcal{E}$  with  $\Phi(\mathcal{F}) \subseteq \mathcal{F}(2+\ell)$ , because the characteristic polynomial of  $B$  is  $\det(tI - B) = (-1)^{r-1}z + t^r$ , which is irreducible in  $\mathbb{C}[z, t]$ .  $\square$

**Remark 5.4.** Assume the genus  $g$  of  $X$  is at least 2 and that  $2 - 2g + \ell < 0$ . Then there exists no semistable meromorphic co-Higgs bundle  $(\mathcal{E}, \Phi)$  with  $\Phi \neq 0$ . Indeed, for any pair  $(\mathcal{E}, \Phi)$ , the map  $\Phi$  would be a non-zero map between two semistable vector bundles with the target having lower slope.

## 6. MODULI OVER PROJECTIVE PLANE

Let  $X = \mathbb{P}^n$  and fix  $\mathcal{D} = \{D\}$  with  $D \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ . Then we have  $T_{\mathbb{P}^n}(-\log \mathcal{D}) \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$  and

$$\Phi = (\varphi_1, \dots, \varphi_n) : \mathcal{E} \rightarrow \mathcal{E} \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$$

with  $\varphi_i : \mathcal{E} \rightarrow \mathcal{E}(1)$  for  $i = 1, \dots, n$ . Assume that  $(\mathcal{E}, \Phi)$  is a semistable co-Higgs bundle of rank  $r$  along  $\mathcal{D}$ . If  $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i)$  is a direct sum of line bundles on  $\mathbb{P}^n$  with  $a_i \geq a_{i+1}$  for all  $i$ , then we get  $a_i \leq a_{i+1} + 1$  for all  $i$  by adapting the proof of [21, Theorem 6.1]. Thus in case  $\text{rank}(\mathcal{E}) = r = 2$ , by a twist we fall into two cases:  $\mathcal{O}_{\mathbb{P}^n}^{\oplus 2}$  or  $\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)$ .

We denote by  $\text{End}_0(\mathcal{E})$  the kernel of the trace map  $\text{End}(\mathcal{E}) \rightarrow \mathcal{O}_X$ , the trace-free part, and then we have

$$\text{End}(\mathcal{E}) \otimes T_X(-\log \mathcal{D}) \cong (\text{End}_0(\mathcal{E}) \otimes T_X(-\log \mathcal{D})) \oplus T_X(-\log \mathcal{D}).$$

Thus any co-Higgs field  $\Phi$  can be decomposed into  $\Phi_1 + \Phi_2$  with  $\Phi_1 \in H^0(\text{End}_0(\mathcal{E}) \otimes T_X(-\log \mathcal{D}))$  and  $\Phi_2 \in H^0(T_X(-\log \mathcal{D}))$ . Note that  $(\mathcal{E}, \Phi)$  is (semi)stable if and only if  $(\mathcal{E}, \Phi_1)$  is (semi)stable. Thus we may pay attention only to trace-free logarithmic co-Higgs bundles. Let us denote by  $\mathbf{M}_{\mathcal{D}}(c_1, c_2)$  the moduli of semistable trace-free  $\mathcal{D}$ -logarithmic co-Higgs bundles of rank two on  $\mathbb{P}^2$  with Chern classes  $(c_1, c_2)$ . In case  $\mathcal{D} = \emptyset$  we simply denote the moduli space by  $\mathbf{M}(c_1, c_2)$ .

**Proposition 6.1.**  $\mathbf{M}_{\mathcal{D}}(-1, 0)$  is isomorphic to the total space of  $\mathcal{O}_{\mathcal{D}}(-2)^{\oplus 6}$ .

*Proof.* By [14, Lemma 3.2]  $\mathcal{E}$  is not semistable for  $(\mathcal{E}, \Phi) \in \mathbf{M}_{\mathcal{D}}(-1, 0)$  and so we get an exact sequence  $0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(t) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(-t-1) \rightarrow 0$  with  $t \geq 0$ . Here  $\Phi(\mathcal{O}_{\mathbb{P}^2}(t)) \subset \mathcal{I}_Z(-t)$  is a non-trivial subsheaf and so we get  $t = 0$  and  $Z = \emptyset$ . Thus we get  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ . Then following the proof of [22, Theorem 5.2] verbatim, we see that

$$\mathbf{M}_{\mathcal{D}}(-1, 0) \cong H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \times (H^0(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \setminus \{0\}) // \mathbb{C}^*,$$

where  $\mathbb{C}^*$  acts on  $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$  with weight  $-2$  and on  $H^0(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \setminus \{0\}$  with weight  $1$ . Thus we get that  $\mathbf{M}_{\mathcal{D}}(-1, 0)$  is isomorphic to the total space of  $\mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2}$ . Indeed,

from the sequence (1) twisted by  $-1$ , we can identify  $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2})$  with  $D$  and so  $\mathbf{M}_{\mathcal{D}}(-1, 0)$  can be obtained by restricting  $\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 6}$  to  $D$  as a closed subscheme of  $\mathbf{M}(-1, 0)$ , which is isomorphic to the total space of  $\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 6}$  (see [22, Theorem 5.2]).  $\square$

Recall in [22, Page 1447] that  $\mathbf{M}(0, 0)$  is 8-dimensional and non-isomorphic to  $\mathbf{M}(-1, 0)$ , with an explicitly described open dense subset. On the contrary to Proposition 6.1, we obtain two-codimensional subspace  $\mathbf{M}_{\mathcal{D}}(0, 0)$  of  $\mathbf{M}(0, 0)$ .

**Proposition 6.2.**  *$\mathbf{M}_{\mathcal{D}}(0, 0)$  contains the total space of  $\mathcal{O}_{\mathbb{P}^2}(-2)$  with the zero section contracted to a point, as an open dense subset.*

*Proof.* Take  $(\mathcal{E}, \Phi) \in \mathbf{M}_{\mathcal{D}}(0, 0)$ . From  $c_2 = c_1^2$ , we get that  $\mathcal{E}$  is not stable and so it fits into the following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(t) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(-t) \rightarrow 0$$

with  $t \geq 0$  and  $\deg(Z) = t^2$ . First assume  $t > 0$ . Since every map  $\mathcal{O}_{\mathbb{P}^2}(t) \rightarrow \mathcal{I}_Z(-t) \otimes T_{\mathbb{P}^2}(-\log \mathcal{D})$  is the zero-map, we get  $\Phi(\mathcal{O}_{\mathbb{P}^2}(t)) \subset \mathcal{O}_{\mathbb{P}^2}(t) \otimes T_{\mathbb{P}^2}(-\log \mathcal{D})$ , contradicting the semistability of  $(\mathcal{E}, \Phi)$ . Now assume  $t = 0$  and so we get  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ . Then we follow the argument in [22, Theorem 5.3] to get the assertion.  $\square$

## 7. COHERENT SYSTEM AND HOLOMORPHIC TRIPLE

If  $\mathcal{F} \subset \mathcal{E}$  is a non-trivial subsheaf, then its saturation  $\tilde{\mathcal{F}}$  is defined to be the maximal subsheaf of  $\mathcal{E}$  containing  $\mathcal{F}$  with  $\text{rank } \tilde{\mathcal{F}} = \text{rank } \mathcal{F}$ ;  $\tilde{\mathcal{F}}$  is the only subsheaf of  $\mathcal{E}$  containing  $\mathcal{F}$  with  $\mathcal{E}/\tilde{\mathcal{F}}$  torsion-free.

**7.1. Coherent system.** Inspired by the theory of coherent systems on smooth algebraic curves in [8], we consider the following definition. Let  $\mathcal{E}$  be a torsion-free sheaf of rank  $r \geq 2$  on  $X$  and  $(\mathcal{E}, \Phi)$  be a  $\mathcal{D}$ -logarithmic co-Higgs structure. Then we define a set

$$\mathcal{S} = \mathcal{S}(\mathcal{E}, \Phi) := \{(\mathcal{F}, \mathcal{G}) \mid 0 \subsetneq \mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{E} \text{ with } \Phi(\mathcal{F}) \subseteq \mathcal{G} \otimes T_X(-\log \mathcal{D})\}.$$

For a fixed real number  $\alpha \geq 0$  and  $(\mathcal{F}, \mathcal{G}) \in \mathcal{S}$ , set

$$\begin{aligned} \mu_{\alpha}(\mathcal{F}, \mathcal{G}) &= \mu(\mathcal{F}) + \alpha \left( \frac{\text{rank } \mathcal{F}}{\text{rank } \mathcal{G}} \right) \\ \mu'_{\alpha}(\mathcal{F}, \mathcal{G}) &= \mu(\mathcal{F}) + \alpha \left( \frac{\text{rank } \mathcal{F}}{\text{rank } \mathcal{F} + \text{rank } \mathcal{G}} \right). \end{aligned}$$

Note that  $\mu_{\alpha}(\mathcal{E}, \mathcal{E}) = \mu(\mathcal{E}) + \alpha$  and  $\mu'_{\alpha}(\mathcal{E}, \mathcal{E}) = \mu(\mathcal{E}) + \alpha/2$ . From now on we use  $\mu_{\alpha}$ , but  $\mu'_{\alpha}$  does the same job. In general, we have  $\mu_{\alpha}(\mathcal{F}, \mathcal{G}) \leq \mu(\mathcal{F}) + \alpha$  for  $(\mathcal{F}, \mathcal{G}) \in \mathcal{S}$  and equality holds if and only if  $\text{rank } \mathcal{F} = \text{rank } \mathcal{G}$ , i.e.  $\mathcal{G}$  is contained in the saturation  $\tilde{\mathcal{F}}$  of  $\mathcal{F}$  in  $\mathcal{E}$ .

**Definition 7.1.** The pair  $(\mathcal{E}, \Phi)$  is said to be  $\mu_{\alpha}$ -stable (resp.  $\mu_{\alpha}$ -semistable) if  $\mu_{\alpha}(\mathcal{F}, \mathcal{G}) < \mu_{\alpha}(\mathcal{E}, \mathcal{E})$  (resp.  $\mu_{\alpha}(\mathcal{F}, \mathcal{G}) \leq \mu_{\alpha}(\mathcal{E}, \mathcal{E})$ ) for all  $(\mathcal{F}, \mathcal{G}) \in \mathcal{S} \setminus \{(\mathcal{E}, \mathcal{E})\}$ . A similar definition is given with  $\mu'_{\alpha}$ .

Note that if  $\mathcal{E}$  is semistable (resp. stable), then a pair  $(\mathcal{E}, \Phi)$  is  $\mu_{\alpha}$ -semistable (resp.  $\mu_{\alpha}$ -stable) for any  $\alpha$  and  $\Phi$ . The converse also holds for  $\Phi = 0$ .

**Remark 7.2.** We have  $\Phi(\mathcal{F}) \subseteq \tilde{\mathcal{G}} \otimes T_X(-\log \mathcal{D})$  for  $(\mathcal{F}, \mathcal{G}) \in \mathcal{S}$  and so to test the  $\mu_\alpha$ -(semi)stability of  $(\mathcal{E}, \Phi)$ , it is sufficient to test the pairs  $(\mathcal{F}, \mathcal{G}) \in \mathcal{S} \setminus \{(\mathcal{E}, \mathcal{E})\}$  with  $\mathcal{G}$  saturated in  $\mathcal{E}$ . Moreover, if  $\mathcal{G}$  is saturated in  $\mathcal{E}$ , then  $\mathcal{G} \otimes T_X(-\log \mathcal{D})$  is saturated in  $\mathcal{E} \otimes T_X(-\log \mathcal{D})$ . Since  $\Phi(\mathcal{F})$  is a subsheaf of  $\Phi(\tilde{\mathcal{F}})$  with the same rank we have  $\Phi(\tilde{\mathcal{F}}) \subseteq \mathcal{G} \otimes T_X(-\log \mathcal{D})$ . So to test the  $\mu_\alpha$ -(semi)stability of  $(\mathcal{E}, \Phi)$  it is sufficient to test the pairs  $(\mathcal{F}, \mathcal{G}) \in \mathcal{S} \setminus \{(\mathcal{E}, \mathcal{E})\}$  with both  $\mathcal{F}$  and  $\mathcal{G}$  saturated in  $\mathcal{E}$ .

**Lemma 7.3.** *If  $(\mathcal{E}, \Phi)$  is not semistable (resp. stable), then it is not  $\mu_\alpha$ -semistable (resp. not  $\mu_\alpha$ -stable) for any  $\alpha$ .*

*Proof.* Take  $\mathcal{F} \subset \mathcal{E}$  such that  $\Phi(\mathcal{F}) \subseteq \mathcal{F} \otimes T_X(-\log \mathcal{D})$  and  $\mu(\mathcal{F}) > \mu(\mathcal{E})$  (resp.  $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$ ). We have  $(\mathcal{F}, \mathcal{F}) \in \mathcal{S}$  and so  $\mu_\alpha(\mathcal{F}, \mathcal{F}) = \mu(\mathcal{F}) + \alpha > (\text{resp. } \geq) \mu(\mathcal{E}) + \alpha = \mu_\alpha(\mathcal{E}, \mathcal{E})$ , proving the assertion.  $\square$

**Remark 7.4.** Lemma 7.3 shows that  $\mu_\alpha$ -stability is stronger than the stability of the pairs  $(\mathcal{E}, \Phi)$  in the sense of [20, 21, 22] and so they form a bounded family if we fix the Chern classes of  $\mathcal{E}$ . However, if  $(\mathcal{E}, \Phi)$  is not  $\mu_\alpha$ -semistable, a pair  $(\mathcal{F}, \mathcal{G}) \in \mathcal{S}$  with  $\mu_\alpha(\mathcal{F}, \mathcal{G}) > \mu(\mathcal{E}) + \alpha$  and maximal  $\mu_\alpha$ -slope may have  $\text{rank}(\mathcal{G}) > \text{rank}(\mathcal{F})$ , i.e.  $\Phi(\mathcal{F}) \not\subseteq \mathcal{F} \otimes T_X(-\log \mathcal{D})$  and so we do not define the Harder-Narasimhan filtration of  $\mu_\alpha$ -unstable pairs  $(\mathcal{E}, \Phi)$ .

**Proposition 7.5.** *Let  $(\mathcal{E}, \Phi)$  be a  $\mathcal{D}$ -logarithmic co-Higgs bundle on  $X$  with  $\mathcal{E}$  not semistable. Then there exist two positive real numbers  $\beta$  and  $\gamma$  such that*

- (i)  $(\mathcal{E}, \Phi)$  is not  $\mu_\alpha$ -semistable for all  $\alpha < \beta$ , and
- (ii) if  $(\mathcal{E}, \Phi)$  is semistable in the sense of Definition 2.2, it is  $\mu_\alpha$ -semistable for all  $\alpha > \gamma$ .

*Proof.* Assume that  $\mathcal{E}$  is not semistable and take a subsheaf  $\mathcal{G}$  with  $\mu(\mathcal{G}) > \mu(\mathcal{E})$ . Note that  $(\mathcal{G}, \mathcal{E}) \in \mathcal{S}$ . Then there exists a real number  $\beta > 0$  such that  $\mu_\alpha(\mathcal{G}, \mathcal{E}) > \mu(\mathcal{E}) + \alpha = \mu_\alpha(\mathcal{E}, \mathcal{E})$  for all  $\alpha$  with  $0 < \alpha < \beta$ . Thus  $(\mathcal{E}, \Phi)$  is not  $\mu_\alpha$ -semistable if  $\alpha < \beta$ .

Now assume that  $\mathcal{E}$  is not semistable, but that  $(\mathcal{E}, \Phi)$  is semistable. Define

$$\Delta = \{\text{the saturated subsheaves } \mathcal{A} \subset \mathcal{E} \mid \mu(\mathcal{A}) > \mu(\mathcal{E})\}.$$

Let  $\mu_{\max}(\mathcal{E})$  be the maximum of the slopes of subsheaves of  $\mathcal{E}$ , which exists as a finite real number by the existence of the Harder-Narasimhan filtration of  $\mathcal{E}$ . Since  $\mathcal{E}$  is not semistable, we have  $\mu_{\max}(\mathcal{E}) > \mu(\mathcal{E})$  and set  $\gamma := r(\mu_{\max}(\mathcal{E}) - \mu(\mathcal{E})) > 0$ . Fix any real number  $\alpha \geq \gamma$ . Now take  $\mathcal{A} \in \Delta$  and set  $s := \text{rank } \mathcal{A}$ . Since  $(\mathcal{E}, \Phi)$  is semistable, we get  $\text{rank } \mathcal{B} > s$ . Thus we have

$$\begin{aligned} \mu_\alpha(\mathcal{A}, \mathcal{B}) &\leq \mu(\mathcal{A}) + \alpha s / (s + 1) \\ &\leq \mu(\mathcal{A}) + \alpha(r - 1) / r \leq \mu_\alpha(\mathcal{E}, \mathcal{E}) \end{aligned}$$

and so  $(\mathcal{E}, \Phi)$  is  $\mu_\alpha$ -semistable for all  $\alpha \geq \gamma$ .  $\square$

**Remark 7.6.** For  $s = 1, \dots, r - 1$ , let  $\Delta_s$  be the set of all  $\mathcal{G} \in \Delta$  with  $\text{rank } \mathcal{G} = s$ . If  $\mu(\mathcal{G}) < \mu_{\max}(\mathcal{E})$  for all  $\mathcal{G} \in \Delta_{r-1}$ , we may use a lower real number instead of  $\gamma$  in the proof of Proposition 7.5.

**Example 7.7.** Let  $X = \mathbb{P}^1$  and take  $\mathcal{D} = \{p\}$  with  $p$  a point. Then we have  $T_{\mathbb{P}^1}(-\log \mathcal{D}) \cong T_{\mathbb{P}^1}(-p) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ . Let  $(\mathcal{E}, \Phi)$  be a semistable  $\mathcal{D}$ -logarithmic co-Higgs bundle of rank  $r \geq 2$  on  $\mathbb{P}^1$  with  $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$  with  $a_1 \geq \dots \geq a_r$  and  $a_i - a_{i+1} \leq 1$  for all  $i = 1, \dots, r - 1$  as in Example 2.4. We assume that  $\mathcal{E}$  is

not semistable, i.e.  $a_r < a_1$ . The value  $\gamma$  in Proposition 7.5 could depend on  $\Phi$ , although it is the same for all general  $\Phi$ . Up to a twist we may assume  $a_1 = 0$ . We have  $\mu(\mathcal{E}) = c_1/r$  with  $c_1 = a_1 + \dots + a_r$ . For each  $s = 1, \dots, r-1$ , set  $b_s = (a_1 + \dots + a_s)/s$  and define

$$\gamma_0 := \max_{1 \leq s \leq r-1} (s+1)(b_s - c_1/r).$$

We have  $\mu(\mathcal{F}) \leq b_s$  for all  $\mathcal{F} \in \Delta_s$  and so  $\mu_\alpha(\mathcal{F}, \mathcal{G}) \leq \mu_\alpha(\mathcal{E}, \mathcal{E})$  for all  $(\mathcal{F}, \mathcal{G})$  with  $\text{rank } \mathcal{F} = s$  and  $\Phi(\mathcal{F}) \not\subseteq \mathcal{F} \otimes T_{\mathbb{P}^1}(-\log \mathcal{D})$ . Hence  $(\mathcal{E}, \Phi)$  is  $\mu_\alpha$ -semistable for all  $\alpha \geq \gamma_0$ .

**Example 7.8.** Similarly as in Example 7.7, we take  $X = \mathbb{P}^1$  and  $\mathcal{D} = \emptyset$ . Then we have  $T_{\mathbb{P}^1}(-\log \mathcal{D}) \cong T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ . We argue as in Example 7.7, except that now we only require that  $a_i - a_{i+1} \leq 2$  for all  $i = 1, \dots, r-1$ .

**Example 7.9.** Take  $X = \mathbb{P}^n$  with  $n \geq 2$  and assume that  $(\mathcal{E}, \Phi)$  is a semistable logarithmic co-Higgs reflexive sheaf of rank two with  $\mathcal{E}$  not semistable. Up to a twist we may assume  $c_1(\mathcal{E}) \in \{-1, 0\}$ . Set  $c_1 := c_1(\mathcal{E})$ . Since  $\mathcal{E}$  is not semistable, we have an exact sequence

$$(6) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(t) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(c_1 - t) \rightarrow 0$$

with either  $Z = \emptyset$  or  $\dim(Z) = n-2$ , and  $t \geq 0$  and  $t > 0$  if  $c_1(\mathcal{E}) = 0$ . Since  $(\mathcal{E}, \Phi)$  is semistable, there is no saturated subsheaf  $\mathcal{A} \subset \mathcal{E}$  of rank one with  $(\mathcal{A}, \mathcal{A}) \in \mathcal{S}$  and  $\mu(\mathcal{A}) > -1$ . Note that  $\mu_\alpha(\mathcal{O}_{\mathbb{P}^n}(t), \mathcal{E}) = t + \alpha/2$  and so  $(\mathcal{E}, \Phi)$  is  $\mu_\alpha$ -stable (resp.  $\mu_\alpha$ -semistable) if and only if  $\alpha > 2t - c_1$  (resp.  $\alpha \geq 2t - c_1$ ).

Now we discuss the existence of such a pair  $(\mathcal{E}, \Phi)$ . Since  $(\mathcal{E}, \Phi)$  is semistable, we should have  $\Phi(\mathcal{O}_{\mathbb{P}^n}(t)) \not\subseteq \mathcal{O}_{\mathbb{P}^n}(t) \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$  and so there is a non-zero map  $\mathcal{O}_{\mathbb{P}^n}(t) \rightarrow \mathcal{I}_Z(c_1 - t) \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})$ . Since  $t > c_1 - t$  and  $h^0(T_{\mathbb{P}^n}(-2)) = 0$ , we get  $t = 0$  and  $c_1 = -1$ . Then we also get  $H^0(\mathcal{I}_Z(-1) \otimes T_{\mathbb{P}^n}(-\log \mathcal{D})) \neq 0$ , which gives restrictions on the choice of  $\mathcal{D}$  and  $Z$ . Assume that  $\mathcal{D} = \{D\}$  with  $D \in |\mathcal{O}_{\mathbb{P}^n}(1)|$  a hyperplane, so that  $T_{\mathbb{P}^n}(-\log \mathcal{D}) \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$ . In this case we get  $Z = \emptyset$  and so  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)$ . See Proposition 6.1 for the associated moduli space in case  $n = 2$ .

**7.2. Holomorphic triple.** We may also consider a holomorphic triple of logarithmic co-Higgs bundles and define its semistability as in [7].

**Definition 7.10.** A holomorphic triple of  $\mathcal{D}$ -logarithmic co-Higgs bundles is a triple  $((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ , where each  $(\mathcal{E}_i, \Phi_i)$  is a  $\mathcal{D}$ -logarithmic co-Higgs sheaf with each  $\mathcal{E}_i$  torsion-free on  $X$  and  $f : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  is a map of sheaves such that  $\Phi_2 \circ f = \hat{f} \circ \Phi_1$ , where  $\hat{f} : \mathcal{E}_1 \otimes T_X(-\log \mathcal{D}) \rightarrow \mathcal{E}_2 \otimes T_X(-\log \mathcal{D})$  is the map induced by  $f$ .

For any real number  $\alpha \geq 0$ , define the  $\nu_\alpha$ -slope of a triple  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$  to be the  $\nu_\alpha$ -slope of the triple  $(\mathcal{E}_1, \mathcal{E}_2, f)$  in the sense of [7], i.e.

$$\nu_\alpha((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f) = \frac{\deg_\alpha(\mathcal{A})}{\text{rank } \mathcal{E}_1 + \text{rank } \mathcal{E}_2},$$

where  $\deg_\alpha(\mathcal{A}) = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_2) + \alpha \text{rank } \mathcal{E}_1$ . A holomorphic subtriple  $\mathcal{B} = ((\mathcal{F}_1, \Psi_1), (\mathcal{F}_2, \Psi_2), g)$  of  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$  is a holomorphic triple with  $\mathcal{F}_i \subset \mathcal{E}_i$ ,  $\Psi_i = \Phi_i|_{\mathcal{F}_i}$  and  $g = f|_{\mathcal{F}_1}$ . Since  $\Phi_i$  is integrable, so is  $\Psi_i$ .

**Remark 7.11.** As before, we may use the slope  $\nu_\alpha$  to define the  $\nu_\alpha$ -(semi)stability for  $\mathcal{D}$ -logarithmic co-Higgs triples. If  $h : \mathcal{A} \rightarrow \mathcal{B}$  is a non-zero map of  $\nu_\alpha$ -semistable holomorphic triples, then we have  $\nu_\alpha(\mathcal{B}) \geq \mu_\alpha(\mathcal{A})$ . Moreover, if  $\mathcal{A}$  is  $\nu_\alpha$ -stable, then either  $\nu_\alpha(\mathcal{B}) > \nu_\alpha(\mathcal{A})$  or  $h$  is injective; in addition, if  $\mathcal{B}$  is also  $\nu_\alpha$ -stable, then  $h$  is an automorphism.

**Remark 7.12.** The degenerate holomorphic triple  $((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), 0)$  with  $f = 0$  is  $\nu_\alpha$ -semistable if and only if  $\alpha = \mu(\mathcal{E}_2) - \mu(\mathcal{E}_1)$  and both  $(\mathcal{E}_i, \Phi_i)$ 's are semistable as in [6, Lemma 3.5]. Moreover such triples are not  $\nu_\alpha$ -stable (see [6, Corollary 3.6]). Note that if  $\Phi_1 = \Phi_2 = 0$ , then we fall into the usual holomorphic triples. We also have an analogous statement for the case  $r_2 = \text{rank } \mathcal{E}_2 = 1$  as in [6, Lemma 3.7].

**Remark 7.13.** For subtriples  $\mathcal{B}$  and  $\mathcal{B}'$  of  $\mathcal{A}$ , we may define their sum and intersection  $\mathcal{B} + \mathcal{B}'$  and  $\mathcal{B} \cap \mathcal{B}'$ ; let  $\mathcal{B} = ((\mathcal{F}_1, \Psi_1), (\mathcal{F}_2, \Psi_2), g)$  and  $\mathcal{B}' = ((\mathcal{F}'_1, \Psi'_1), (\mathcal{F}'_2, \Psi'_2), g)$ . Then we may use  $\mathcal{F}_i + \mathcal{F}'_i$  and  $\mathcal{F}_i \cap \mathcal{F}'_i$  with the restrictions of  $\Phi_i$  and  $f$  to them. Now call  $\tilde{\mathcal{F}}_i$  the saturation of  $\mathcal{F}_i$  in  $\mathcal{E}_i$ . Since  $\tilde{\mathcal{F}}_i \otimes T_X(-\log \mathcal{D})$  is saturated in  $\mathcal{E}_i \otimes T_X(-\log \mathcal{D})$ , we have  $\Phi_i(\tilde{\mathcal{F}}_i) \subseteq \tilde{\mathcal{F}}_i \otimes T_X(-\log \mathcal{D})$ . Since  $f(\mathcal{F}_1) \subseteq \tilde{\mathcal{F}}_2$ , we have  $f(\tilde{\mathcal{F}}_1) \subseteq \tilde{\mathcal{F}}_2$  and so we may also define the saturation  $\tilde{\mathcal{B}}$  of  $\mathcal{B}$  with  $\nu_\alpha(\tilde{\mathcal{B}}) \geq \nu_\alpha(\mathcal{B})$ .

Fix  $\alpha \in \mathbb{R}_{>0}$  and let  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$  be a holomorphic triple. We define  $\beta(\mathcal{A})$  to be the maximum of the set of the  $\nu_\alpha$ -slopes of all subtriples of  $\mathcal{A}$  and let

$$\mathbb{B} := \{\mathcal{B} \subseteq \mathcal{A} \mid \nu_\alpha(\mathcal{B}) = \beta(\mathcal{A})\}.$$

**Lemma 7.14.** *The set of the  $\nu_\alpha$ -slopes of all subtriples of  $\mathcal{A}$  is upper bounded and so  $\beta(\mathcal{A})$  exists. Moreover, the set  $\mathbb{B}$  has a unique maximal element*

*Proof.* The ranks of any non-zero subsheaf of  $\mathcal{E}_i$  is upper bounded by  $r_i := \text{rank } \mathcal{E}_i$  and lower bounded by 1. The existence of the Harder-Narasimhan filtration of  $\mathcal{E}_i$  gives the existence of positive rational numbers  $\gamma_i$  with denominators between 1 and  $r_i$  such that  $\mu(\mathcal{F}_i) \leq \gamma_i$  for all non-zero subsheaves  $\mathcal{F}_i$  of  $\mathcal{E}_i$ . We may use the definition of  $\nu_\alpha$ -slope to get an upper-bound for the  $\nu_\alpha$ -slopes of the subtriples of  $\mathcal{A}$ . There are only finitely many possible  $\nu_\alpha$ -slopes greater than  $\nu_\alpha(\mathcal{A})$ , because the ranks are upper and lower bounded and each  $\deg(\mathcal{G})$  for a subsheaf  $\mathcal{G}$  of  $\mathcal{E}_i$  is an integer, upper bounded by  $\max\{r_1\mu(\mathcal{E}_1), r_2\mu(\mathcal{E}_2)\}$ . Thus the set of the  $\nu_\alpha$ -slopes of all subtriples of  $\mathcal{A}$  has a maximum  $\beta(\mathcal{A})$ .

If  $\nu_\alpha(\mathcal{A}) = \beta(\mathcal{A})$ , then  $\mathcal{A}$  itself is the maximum element of  $\mathbb{B}$ . Now assume  $\nu_\alpha(\mathcal{A}) > \delta$  and that there are  $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{B}$  with each  $\mathcal{B}_i$  maximal and  $\mathcal{B}_1 \neq \mathcal{B}_2$ . Since  $\mathcal{B}_i$  is maximal, it is saturated and so  $\mathcal{A}_i := \mathcal{A}/\mathcal{B}_i$  is a holomorphic triple for each  $i$ . Since  $\mathcal{B}_2 \neq \mathcal{B}_1$ , the inclusion  $\mathcal{B}_2 \subset \mathcal{A}$  induces a non-zero map  $u : \mathcal{B}_2 \rightarrow \mathcal{A}/\mathcal{B}_1$ . Since  $\nu_\alpha(\ker(u)) \leq \beta(\mathcal{A})$  if  $u$  is not injective, we have  $\nu_\alpha(u(\mathcal{A}/\mathcal{B}_1)) \geq \beta(\mathcal{A})$ . Thus we get  $\nu_\alpha(\mathcal{B}_1 + \mathcal{B}_2) \geq \beta(\mathcal{A})$ , contradicting the maximality of  $\mathcal{B}_1$  and the assumption  $\mathcal{B}_2 \neq \mathcal{B}_1$ .  $\square$

Assume that  $\mathcal{A}$  is not  $\nu_\alpha$ -semistable. By Lemma 7.14 there is a subtriple  $D(\mathcal{A}) = ((\mathcal{F}_1, \Psi_1), (\mathcal{F}_2, \Psi_2), g) \in \mathbb{B}$  such that every  $\mathcal{G} \in \mathbb{B}$  is a subtriple of  $D(\mathcal{A})$  and each  $\mathcal{F}_i$  is saturated in  $\mathcal{E}_i$ . Note that  $D(\mathcal{A})$  is  $\nu_\alpha$ -semistable. Since  $\mathcal{F}_i$  is saturated in  $\mathcal{E}_i$  and  $\Psi_i = \Phi_i|_{\mathcal{F}_i}$  for each  $i$ ,  $\Phi_i$  induces a co-Higgs field  $\tau_i : \mathcal{E}_i/\mathcal{F}_i \rightarrow (\mathcal{E}_i/\mathcal{F}_i) \otimes T_X(-\log \mathcal{D})$ . Since  $\Phi_i$  is integrable, so is  $\tau_i$ . Since  $g = f|_{\mathcal{F}_1}$ ,  $f$  induces a map  $f' : \mathcal{E}_1/\mathcal{F}_1 \rightarrow \mathcal{E}_2/\mathcal{F}_2$  such that  $\mathcal{A}/D(\mathcal{A}) := ((\mathcal{E}_1/\mathcal{F}_1, \tau_1), (\mathcal{E}_2/\mathcal{F}_2, \tau_2), f')$  is a holomorphic triple. Now we

may check that each subtriple of  $\mathcal{A}/D(\mathcal{A})$  has  $\nu_\alpha$ -slope less than  $\beta(\mathcal{A})$  and so  $D(\mathcal{A})$  defines the first step of the Harder-Narasimhan filtration of  $\mathcal{A}$ . The iteration of this process allows us to have the Harder-Narasimhan filtration of  $\mathcal{A}$  with respect to  $\nu_\alpha$ .

**Corollary 7.15.** *Any holomorphic triple admits the Harder-Narasimhan filtration with respect to  $\nu_\alpha$ -slope.*

**Remark 7.16.** Let  $Z$  denote a projective completion of  $T_X(-\log \mathcal{D})$ , e.g.  $Z = \mathbb{P}(\mathcal{O}_X \oplus T_X(-\log \mathcal{D}))$ , and call  $D_\infty := Z \setminus T_X(-\log \mathcal{D})$  the divisor at infinity. By [24, Lemma 6.8] a co-Higgs sheaf  $(\mathcal{E}, \Phi)$  on  $X$  is the same thing as a coherent sheaf  $\mathcal{E}_Z$  with  $\text{Supp}(\mathcal{E}_Z) \cap D_\infty = \emptyset$ . Due to [24, Corollary 6.9] we may interpret a  $\nu_\alpha$ -semistable holomorphic triple of logarithmic co-Higgs bundles on  $X$  as a  $\nu_\alpha$ -semistable holomorphic triple of vector bundles on  $Z$  with support not intersecting  $D_\infty$  as in [7].

Based on Remark 7.16, we may consider a  $\nu_\alpha$ -semistable triple of  $\mathcal{D}$ -logarithmic co-Higgs sheaves as a  $\nu_\alpha$ -semistable quiver sheaf for the quiver  $\overset{1}{\circ} \longrightarrow \overset{2}{\circ}$  on  $Z$  with empty intersection with  $D_\infty$ . This interpretation ensures the existence of moduli space of  $\nu_\alpha$ -stable triples of  $\mathcal{D}$ -logarithmic co-Higgs sheaves on  $X$ , say  $\mathbf{M}_{\mathcal{D}, \alpha}(r_1, r_2, d_1, d_2)$  with  $(r_i, d_i)$  a pair of rank and degree of the  $i^{\text{th}}$ -factor of the triples; indeed we may consider Gieseker-type semistability of quiver sheaves to produce the moduli space as in [23]. As noticed in [23, Remark in page 17], the  $\nu_\alpha$ -stability implies the Gieseker-type stability and so  $\mathbf{M}_{\mathcal{D}, \alpha}(r_1, r_2, d_1, d_2)$  can be considered as a quasi-projective subvariety of the one in [23]. Now let us define

$$\alpha_m := \mu(\mathcal{E}_2) - \mu(\mathcal{E}_1), \quad \alpha_M := \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) (\mu(\mathcal{E}_2) - \mu(\mathcal{E}_1))$$

for  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$  as in [19]. Then we have

**Proposition 7.17.** [7, Proposition 2.2] *If  $\alpha > \alpha_M$  with  $\text{rank } \mathcal{E}_1 \neq \text{rank } \mathcal{E}_2$  or  $\alpha < \alpha_m$ , then there exists no  $\nu_\alpha$ -semistable triple of  $\mathcal{D}$ -logarithmic co-Higgs sheaves.*

*Proof.* Due to [24, Corollary 6.9], it is sufficient to check the assertion for  $\nu_\alpha$ -semistability for a triple of coherent sheaves on  $Z$ . While the proof of [7, Proposition 2.2] is for curves, the proof is numerical involving rank and degree with respect to a fixed ample line bundle so that it works also for  $Z$ .  $\square$

From now on we assume that  $X$  is a smooth projective curve of genus  $g$  and let  $\mathcal{D} = \{p_1, \dots, p_m\}$  be a set of  $m$  distinct points on  $X$ . Take  $g \in \{0, 1\}$  and assume that  $T_X(-\log \mathcal{D}) \cong \mathcal{O}_X$ , i.e.  $(g, m) \in \{(0, 2), (1, 0)\}$ . For any triple  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$  and  $c \in \mathbb{C}$ , set

$$(7) \quad \mathcal{A}_c := ((\mathcal{E}_1, \Phi_1 - c \cdot \text{Id}_{\mathcal{E}_1}), (\mathcal{E}_2, \Phi_2 - c \cdot \text{Id}_{\mathcal{E}_2}), f)$$

and then  $\mathcal{A}_c$  is also a triple. In particular, if  $\mathcal{E}_1 \cong \mathcal{E}_2$  and  $f \cong c \cdot \text{Id}_{\mathcal{E}_1}$ , then the study of the  $\nu_\alpha$ -(semi)stability of  $\mathcal{A}$  is reduced to the known case  $f = 0$ .

**Remark 7.18.** Assume that  $f$  is not injective. Since  $\hat{f} \circ \Phi_1 = \Phi_2 \circ f$ , we have  $\Phi_1(\ker(f)) \subseteq \ker(\hat{f})$  and  $\mathcal{B} := ((\ker(f), \Phi_1|_{\ker(f)}), (0, 0), 0)$  is a subtriple of  $\mathcal{A}$ . Set  $\rho := \text{rank}(\ker(f))$  and  $\delta := \deg(\ker(f))$ . If we have

$$\nu_\alpha(\mathcal{B}) = \delta/\rho + \alpha > \frac{r_1\alpha + d_1 + d_2}{r_1 + r_2},$$

then  $\mathcal{A}$  would not be  $\nu_\alpha$ -semistable.

**Remark 7.19.** For any triple  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$ , we get a dual triple  $\mathcal{A}^\vee = ((\mathcal{E}_2^\vee, \Phi_2^\vee), (\mathcal{E}_1^\vee, \Phi_1^\vee), f^\vee)$ , where  $\Phi_i^\vee$  and  $f^\vee$  are the transpose of  $\Phi_i$  and  $f$ , respectively. Then  $\mathcal{A}$  is  $\nu_\alpha$ -(semi)stable if and only if  $\mathcal{A}^\vee$  is  $\nu_\alpha$ -(semi)stable (see [6, Proposition 3.16]).

**Remark 7.20.** Assume  $(g, m) = (1, 0)$  and take a triple  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$  with each  $\mathcal{E}_i$  simple. By Atiyah's classification of vector bundles on elliptic curves, the simpleness of  $\mathcal{E}_i$  is equivalent to its stability and also equivalent to its indecomposability and with degree and rank coprime. Then each  $\Phi_i$  is the multiplication by a constant, say  $c_i$ . We get that the two triples  $\mathcal{A}$  and  $((\mathcal{E}_1, 0), (\mathcal{E}_2, 0), f)$  share the same subtriples and so these two triples are  $\nu_\alpha$ -(semi)stable for the same  $\alpha$  simultaneously. There is a good description of this case in [18, Section 7].

Now we suggest some general description on  $\nu_\alpha$ -(semi)stable triples on  $X$  in case of  $r_1 = r_2 = 2$  from (a)  $\sim$  (c) below; we exclude the case described in Remark 7.20 and silently use Remark 7.19 to get a shorter list. In some case we stop after reducing to a case with  $f$  not injective, i.e. to a case in which  $\mathcal{A}$  is not  $\nu_\alpha$ -semistable for  $\alpha \gg 0$  (see Remark 7.18).

(a) Assume  $r_1 = r_2 = 2$  and that at least one of  $\mathcal{E}_i$  is not semistable, say  $\mathcal{E}_1$ . Then, due to Segre-Grothendieck theorem and Atiyah's classification of vector bundles on elliptic curves, we have  $\mathcal{E}_1 \cong \mathcal{L}_1 \oplus \mathcal{R}_1$  with  $\deg(\mathcal{L}_1) > \deg(\mathcal{R}_1)$  and  $\mathcal{E}_2 \cong \mathcal{L}_2 \oplus \mathcal{R}_2$  with  $\deg(\mathcal{L}_2) \geq \deg(\mathcal{R}_2)$ , or  $g = 1$  and  $\mathcal{E}_2$  is a non-zero extension of the line bundle  $\mathcal{L}_2$  by itself; in the latter case we put  $\mathcal{R}_2 := \mathcal{L}_2$ . If  $\mathcal{E}_2$  is indecomposable, then it has a unique line bundle isomorphic to  $\mathcal{L}_2$  and so  $\Phi_2(\mathcal{L}_2) \subseteq \mathcal{L}_2$ . We have

$$\nu_\alpha(\mathcal{A}) = \alpha/2 + (\deg(\mathcal{L}_1) + \deg(\mathcal{L}_2) + \deg(\mathcal{R}_1) + \deg(\mathcal{R}_2))/4.$$

The map  $\Phi_i : \mathcal{E}_i \rightarrow \mathcal{E}_i$  induces a map  $\Phi_{i|\mathcal{L}_i} : \mathcal{L}_i \rightarrow \mathcal{L}_i$ , which is induced by the multiplication by a constant, say  $c_i$ . Then we get two triples  $\mathcal{A}_{c_i}$  for  $i = 1, 2$ . Since  $\mathcal{A}_{c_2}$  is a triple, we get  $f(\mathcal{L}_1) \subseteq \mathcal{L}_2$  and so we may define a subtriple  $\mathcal{A}_1 := ((\mathcal{L}_1, \Phi_{1|\mathcal{L}_1}), (\mathcal{L}_2, \Phi_{2|\mathcal{L}_2}), f|_{\mathcal{L}_1})$  with

$$\begin{aligned} \nu_\alpha(\mathcal{A}_1) &= \alpha + (\deg(\mathcal{L}_1) + \deg(\mathcal{L}_2))/2 \\ &> \alpha/2 + (\deg(\mathcal{L}_1) + \deg(\mathcal{L}_2) + \deg(\mathcal{R}_1) + \deg(\mathcal{R}_2))/4 = \nu_\alpha(\mathcal{A}), \end{aligned}$$

which implies that  $\mathcal{A}$  is not  $\nu_\alpha$ -semistable.

(b) From now on we assume that  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are semistable. We also assume that  $f$  is non-zero so that  $\mu(\mathcal{E}_1) \leq \mu(\mathcal{E}_2)$ . We are in a case with  $r_1 = r_2 = 2$  and we look at a proper subtriple  $\mathcal{B} = ((\mathcal{F}_1, \Phi_{1|\mathcal{F}_1}), (\mathcal{F}_2, \Phi_{2|\mathcal{F}_2}), f|_{\mathcal{F}_1})$  with maximal  $\nu_\alpha(\mathcal{B})$ . In particular, each  $\mathcal{F}_i$  is saturated in  $\mathcal{E}_i$ , i.e. either  $\mathcal{F}_i = \mathcal{E}_i$  or  $\mathcal{F}_i = 0$  or  $\mathcal{E}_i/\mathcal{F}_i$  is a line bundle. Set  $s_i := \text{rank}(\mathcal{F}_i)$  and then we have  $1 \leq s_1 + s_2 \leq 3$ . If  $s_2 = 2$ , i.e.  $\mathcal{F}_2 = \mathcal{E}_2$ , then we have  $\nu_\alpha(\mathcal{B}) < \nu_\alpha(\mathcal{A})$  for all  $\alpha > 0$ , because  $\mathcal{E}_1$  is semistable and  $\mu(\mathcal{E}_1) \leq \mu(\mathcal{E}_2)$ . If  $s_2 = 0$ , then  $f$  is not injective. If  $s_1 = 0$  we just exclude the case  $\alpha \leq \alpha_m$  with subtriple  $((0, 0), (\mathcal{E}_2, \Phi_2), 0)$ . In the case  $s_1 = s_2 = 1$  we know that  $\nu_\alpha(\mathcal{B}) \leq \nu_\alpha(\mathcal{A})$  and that equality holds if and only if both  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are strictly semistable and each  $\mathcal{F}_i$  is a line subbundle of  $\mathcal{E}_i$  with maximal degree. Note that the injectivity of  $f$  implies  $s_1 \leq s_2$ . Thus when  $f$  is injective, it is sufficient to test the case  $s_1 = s_2 = 1$ . Then we have the following, when  $f$  is injective.

- If  $\alpha > \alpha_m$  and at least one of  $\mathcal{E}_i$ 's is stable, then  $\mathcal{A}$  is  $\nu_\alpha$ -stable
- If  $\alpha \geq \alpha_m$  and  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are semistable, then  $\mathcal{A}$  is  $\nu_\alpha$ -semistable.



- If  $\alpha > \alpha_m$  and  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are strictly semistable, then  $\mathcal{A}$  is strictly  $\nu_\alpha$ -semistable if and only if there are maximal degree line bundles  $\mathcal{L}_i \subset \mathcal{E}_i$  such that  $\Phi_i(\mathcal{L}_i) \subseteq \mathcal{L}_i$  for each  $i$  and  $f(\mathcal{L}_1) \subseteq \mathcal{L}_2$ .

**Lemma 7.21.** *For a general map  $f : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  with  $\mathcal{E}_i := \mathcal{O}_{\mathbb{P}^1}(a_i)^{\oplus 2}$  and  $a_2 \geq a_1 + 2$ , there exists no subsheaf  $\mathcal{O}_{\mathbb{P}^1}(a_1) \subset \mathcal{E}_1$  such that the saturation of its image in  $\mathcal{E}_2$  is a line bundle isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(a_2)$ .*

*Proof.* Up to a twist we may assume that  $a_1 = 0$ . If we fix homogeneous coordinates  $x_0, x_1$  on  $\mathbb{P}^1$ , then the map  $f$  is induced by two forms  $u(x_0, x_1)$  and  $v(x_0, x_1)$  of degree  $a_2$ . Then it is sufficient to prove that there is no point  $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$  with which  $au(x_0, x_1) + bv(x_0, x_1)$  is either identically zero or with a zero of multiplicity  $a_2$ . This is true for general  $u(x_0, x_1)$  and  $v(x_0, x_1)$ , e.g. we may take  $u(x_0, x_1) = x_0^{a_2} + x_0x_1^{a_2-1}$  and  $v(x_0, x_1) = x_0x_1^{a_2-1} + x_1^{a_2}$ .  $\square$

The next is an analogue of Lemma 7.21 for elliptic curves.

**Lemma 7.22.** *Let  $X$  be an elliptic curve with two line bundles  $\mathcal{L}_i$  for  $i = 1, 2$  such that  $\deg(\mathcal{L}_2) \geq \deg(\mathcal{L}_1) + 4$ . For a general map  $f : \mathcal{L}_1^{\oplus 2} \rightarrow \mathcal{L}_2^{\oplus 2}$ , there is no subsheaf  $\mathcal{L}_1 \subset \mathcal{L}_1^{\oplus 2}$  such that the saturation of its image in  $\mathcal{L}_2^{\oplus 2}$  is isomorphic to  $\mathcal{L}_2$ .*

*Proof.* It is sufficient to find an injective map  $h : \mathcal{L}_1^{\oplus 2} \rightarrow \mathcal{L}_2^{\oplus 2}$  for which no subsheaf  $\mathcal{L}_1 \subset \mathcal{L}_1^{\oplus 2}$  has its image under  $h$  whose saturation in  $\mathcal{L}_2^{\oplus 2}$  is isomorphic to  $\mathcal{L}_2$ . Up to a twist we may assume  $\mathcal{L}_1 \cong \mathcal{O}_X$  and so  $l := \deg(\mathcal{L}_2) \geq 4$ . First assume  $l = 4$  and write  $\mathcal{L}_2 \cong \mathcal{M}^{\otimes 2}$  with  $\deg(\mathcal{M}) = 2$ . If  $\varphi : X \rightarrow \mathbb{P}^1$  be a morphism of degree two, induced by  $|\mathcal{M}|$ , then we may set  $h := \varphi^*(h_1)$  for a general  $h_1 : \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}$  with Lemma 7.21 applied to  $h_1$ .

Now assume  $l \geq 5$  and fix an effective divisor  $D \subset X$  of degree  $l - 4$ . Then we may take as  $h$  the composition of a general map  $\mathcal{O}_X^{\oplus 2} \rightarrow \mathcal{L}_2(-D)^{\oplus 2}$  with the map  $\mathcal{L}_2(-D)^{\oplus 2} \rightarrow \mathcal{L}_2^{\oplus 2}$  obtained by twisting with  $\mathcal{O}_X(D)$ .  $\square$

**Remark 7.23.** Let  $\mathcal{D}$  be an arrangement with  $T_X(-\log \mathcal{D}) \cong \mathcal{O}_X$  on  $X$  with arbitrary dimension. For two line bundles  $\mathcal{L}_1$  and  $\mathcal{L}_2$  with  $\mathcal{L}_2 \otimes \mathcal{L}_1^\vee$  globally generated, set a triple  $\mathcal{B} = ((\mathcal{E}_1, 0), (\mathcal{E}_2, 0), f)$  with  $\mathcal{E}_i \cong \mathcal{L}_i^{\oplus r}$  and  $f$  injective. As in (7) we may generate other triples  $\mathcal{B}_c$  for each  $c \in \mathbb{C}$ , but often there are no other  $\mathcal{D}$ -logarithmic co-Higgs triples with  $\mathcal{B}$  as the associated triple of vector bundles. For example, assume  $X$  is a smooth projective curve of genus  $g \in \{0, 1\}$ . For a fixed co-Higgs field  $\Phi_1 : \mathcal{E}_1 \rightarrow \mathcal{E}_1$  with the associated  $(r \times r)$ -matrix  $A_1$  of constants, we are looking for  $f$  and  $\Phi_2 : \mathcal{E}_2 \rightarrow \mathcal{E}_2$  with the associated matrix  $A_2$  such that  $\mathcal{A} = ((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2), f)$  is a  $\mathcal{D}$ -logarithmic co-Higgs triple. Let  $M$  be the  $(r \times r)$ -matrix with coefficient in  $H^0(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)$  associated to  $f$ . Then we need  $A_2$  and  $M$  such that  $A_2 M = M A_1$ . Assume that  $A_1$  has a unique Jordan block. If  $\mathcal{L}_1 \cong \mathcal{L}_2$  and  $M$  is general, then we get a  $\mathcal{D}$ -logarithmic co-Higgs triple if and only if  $A_2$  is a polynomial in  $A_1$ . If  $\mathcal{L}_1 \not\cong \mathcal{L}_2$  and  $f$  is general, then there is no such  $A_2$ . We check this for the case  $r = 2$  and the general case can be shown similarly. With no loss of generality we may assume that the unique eigenvalue of  $A_1$  is zero. Assume the existence of  $f$  and  $\Phi_2$  with associated  $M$  and  $A_2$ . We have  $\ker(\Phi_1) \cong \mathcal{L}_1$  and  $f(\ker(\Phi_1)) \subseteq \ker(\Phi_2)$ . Thus we get that  $f(\mathcal{L}_1)$  has  $\ker(\Phi_2) \cong \mathcal{L}_2$  as its saturation, contradicting Lemmas 7.21 and 7.22 for a general  $f$ .

**Remark 7.24.** In the same way as in [1] one can define  $\mathcal{D}$ -logarithmic co-Higgs holomorphic chains with parameters, but if the maps are general, then very few logarithmic co-Higgs fields  $\Phi_i$  are allowed.

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